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A PROBABILISTIC METHOD OF DESIGNING SPECIFIED RELIABILITIES INTO MECHANICAL COMPONENTS WITH TIME DEPENDENT STRESS AND STRENGTH DISTRIBUTIONS

by

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This report is being made available in its present form with the hope it will be of sufficient value to justify its announcement and distribution.

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LIST OF SYMBOLS

Symbol	Description
A	Area, inches^2
A_i	Area - inner diameter, inches^2
A_o	Area - outer diameter, inches^2
B_1 (a, b)	Type one Beta distribution with parameters a and b
B_2 (a, b)	Type two Beta distribution with parameters a and b
C_t	Fatigue factor
c	Distance from center to outer fiber, inches
D	Major diameter, inches
d	Minor diameter, inches
d_i	Inner diameter, inches
d_o	Outer diameter, inches
d_s	Shaft diameter, inches
E	Modulus of elasticity, pounds per square inch
$E\left[f(x)\right]$	Mathematical expectation of f(x)
e	2.7183
F	Force, pounds
F(x), G(x), ···	Cumulative distribution function of x
F(u)	Fourier transform of u
$F^{-1}(z)$	Inverse Fourier transform

Symbol	Description
$f(x)$, $g(x)$	Probability density function of x
$f(S)$	Strength probability density function
$f(s)$	Stress probability density function
$G(\beta, \eta)$	Gamma p. d. f. with parameters β and η
$g_1(y_1)$	Marginal p. d. f. of y_1
h	Height, inches
h_i	Hole height, inches
h_o	Overall height, inches
I	Moment of inertia, inches
J	Polar moment of inertia, inches
J	Jacobian of inverse transformation
K_b	Theoretical stress concentration factor in bending
K_f	Fatigue stress concentration factor
K_t	Theoretical stress concentration factor
k_a	Modifying factor for surface
k_b	Modifying factor for size
k_c	Modifying factor for reliability
k_d	Modifying factor for temperature
k_e	Modifying factor for stress concentration
k_f	Modifying factor for miscellaneous effects
$K_o(x)$	Bessel function of second kind

Symbol	Description
$L(\mu, \sigma)$	Lognormal distribution with parameters μ and σ
ℓ	Characteristic length, inches
M	Moment, inches-pounds
$M(f(x))$	Mellin transform of f(x)
$M[f(x) \mid \xi]$	Mellin transform of f(x) with s replaced by ξ
$N(\mu, \sigma)$	Normal p. d. f. with parameters μ and σ
n	Revolutions per minute
P. F.	Probability of failure
Pr (x)	Probability of x
Q(t)	Unreliability as a function of time
q	Notch sensitivity factor
R	Radius, inches
R(t)	Reliability as a function of time
r	Radius, inches
R_i	Resistance, ohms, i = 1, 2, 3, 4
S	Strength
$\bar{S}$	Mean strength
S_e	Theoretical endurance limit, thousand pounds per square inch
S'_e	Endurance limit corrected with service factors, thousand pounds per square inch
S	Material strength in bending, thousand pounds per square inch
S_s	Torsional modulus of rupture, thousand pounds per square inch

Symbol	Description
S_{se}	Shear endurance limit
S_u	Ultimate strength, thousand pounds per square inch
s	Stress, pounds per square inch
$\bar{s}$	Mean stress, pounds per square inch
S_y	Yield strength, pounds per square inch
s_a	Alternating stress, pounds per square inch
s_m	Mean stress, pounds per square inch
T	Torque, inches-pounds
t	Age or mission time, hours
$W(\gamma, \beta, \eta)$	Weibull p. d. f. with parameters γ, β, and η
$X, Y, Z, \cdots$	Random variables, general
$x, y, z, \cdots$	Outcomes of random events

GREEK SYMBOLS

β	Weibull shape parameter, Gamma scale parameter
$\Gamma(X)$	Gamma function of X
γ	Weibull location parameter, Gamma shape parameter
ϵ	Strain, micro inches/inch
$A \epsilon B$	Set A is an element of set B
η	Weibull scale parameter
$\lambda(\alpha)$	Failure rate
$\lambda(t)$	Failure rate as a function of time, failures/hour

Symbol	Description
μ	Mean of distribution
μ	Poisson's ratio
γ	Safety factor, distribution
$\prod_{1}^{n}$	Product from 1 to n
$\sum_{1}^{n}$	Sum from 1 to n
σ	Standard deviation of distribution
σ	Stress, pounds per square inch
σ_i	Stress, pounds per square inch, $i = 1,2$
τ	Shear stress, pounds per square inch
τ_d	Shear stress in shaft, pounds per square inch
$\tau_{d_{ALL}}$	Allowable shear stress in shaft, pounds per square inch
$\Phi(t)$	Characteristic function of t
$\chi^2(r)$	Chi - Square distribution with r degrees of freedom

LIST OF ABBREVIATIONS

ASME	American Society of Mechanical Engineers
ASTM	American Society for Testing Materials
BR	Basic Rating
DOD	Department of Defense
D/N	Drawing Number
FS	Factor of Safety
HP	Horsepower
HSS	High Speed Shaft
in.	Inch
LSS	Low Speed Shaft
lb.	Pound
micro	10^{-6}
NASA	National Aeronautics and Space Administration
psi	Pounds per square inch
rpm	Revolutions per minute
SAE	Society of Automotive Engineers
SCF	Stress concentration factor
SF	Service factor
SM	Safety margin
S-N	Stress-cycle
SNAP	Space Nuclear Auxiliary Power
VMHG	Von Mises-Hencky-Goodman
v	Volt

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A PROBABILISTIC METHOD OF DESIGNING SPECIFIED

RELIABILITIES INTO MECHANICAL COMPONENTS WITH

TIME DEPENDENT STRESS AND STRENGTH DISTRIBUTIONS

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SUMMARY

A methodology is presented, for the first year's effort, which enables an engineer to design a specified reliability into a component which has time dependent strength distributions. The case of a component subjected to combined-stress fatigue is treated specifically. The theories of material failure which apply to components in fatigue are reviewed, and the Von Mises theory of failure (distortion energy) is selected for primary use. The maximum shear stress theory of failure is used as an alternate. A large number of examples are given which illustrate the design-by-reliability methodology. Basically, the methodology consists of the following three steps:

1. Determining the failure-governing strength.

2. Determining the failure-governing stress.

3. Bridging the gap by reliability theory.

A thorough discussion of mathematical methods used in problems of functions of random variables is included. The methods discussed are:

1. Algebra of Normal Functions.

2. Change of Variable.

3. Moment Generating Function.

4. Fourier Transform, Convolution, and Inversion.

5. Mellin Transform, Convolution, and Inversion.

6. Characteristic Function.

7. Cumulative Distribution Function.

8. Monte Carlo.

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Definitions of appropriate distributions are given, and a discussion of their application to structural reliability is also given. Results of applying the mathematical techniques to these distributions are presented. The results are useful in the area of structural reliability.

Methods for determining the distributions of failure-governing stress and strength are given. This includes the determination of the basic stress and strength distributions, the distributions of influencing factors, and methods for combining these into the final failure-governing distributions. Analytical, numerical, and experimental methods are discussed.

The methods for bridging the gap by reliability theory once the failure-governing stress and strength distributions have been determined are given. This includes analytical and numerical methods for both normal and non-normal distributions.

Also described is the design, fabrication, and operation of fatigue testing machines for reliability research. These machines utilize the four-square principle to test relatively large specimens under reversed bending and steady torque. These machines are providing valuable data for combined-stress fatigue and for the generation of experimentally-determined strength surfaces (three-dimensional Goodman diagrams) for the design-by-reliability methodology.

A test program now being conducted at The University of Arizona is described in detail. This program is providing data for the demonstration of the design-by-reliability methodology.

Recommendations for further research in this area are given. References are given at the end of each Section.

INTRODUCTION

Prior to this research effort, the basic methodology for designing reliability into mechanical components by consideration of the interference of their stress-strength distributions was discussed by Kececioglu and Cormier (1)*. Included in this paper was a discussion of Monte Carlo techniques for determining stress and strength distributions, given the distributions of the factors affecting them.

Freudenthal (2) wrote a paper in which structural unreliability was considered to be the probability, or risk, of failure. The safety factor was shown to be a distribution function which is the quotient of the strength to the stress, where both strength and stress are considered as statistical variables. Freudenthal, Garrelts, and Shinozuka (3) prepared a comprehensive report along the same lines which discussed in more detail the mathematical techniques required, the appropriate statistical distributions involved, and problems which remained to be solved. Several example problems in structural reliability were worked out, an extensive bibliography was given. These efforts concentrated on simple fatigue and structural reliability.

The Battelle Memorial Institute, and its Mechanical Reliability Research Center presented studies (4), (5) which described some of the fundamental problems in mechanical reliability, and suggested methods for their solution.

Mittenbergs (4) discussed the fundamental aspects of reliability engineering as they pertain to mechanical devices. He stated that the failure modes of mechanical elements were basically:

1. Deformation.

2. Fracture.

3. Instability.

He also asserted that many factors combine to determine the reliability of a mechanical part under such failure modes. The interaction of strength and load distributions was discussed. The Sixth Progress Report of the Mechanical Reliability Research Center (5) summarized a two-year research effort. This extensive research effort contained a thorough discussion of mechanical reliability, and attempted to quantify the relationships of various factors on such phenomena as creep and fatigue. An extensive bibliography was included.

*Numbers in parentheses refer to References at the end of the Introduction.

The IIT Research Institute conducted a program in "Methods for Prediction of Electro-Mechanical Systems Reliability" (6). The program was concerned with three major areas:

1. The study of prime mechanisms of failure in mechanical design. Specific items included fatigue, surface fatigue, wear, creep, and corrosion.

2. The application of failure mechanism and design information for the reliability evaluation of specific mechanical parts. Parts included were gears, bearings, springs, and shafts.

3. The determination of mechanical system reliability in terms of individual part reliability figures.

A paper by G. Reethof, M. J. Bratt, and G. W. Weber of the Large Jet Engine Department, General Electric Company, entitled "A Model for Time Varying and Interfering Stress-Strength Probability Degradation," (7) provided a computer approach towards the solution of the time variant strength distribution case only and did not provide a complete solution to it.

The above works provided some interesting and valuable contributions to the problem of designing specified reliabilities into mechanical components. However, a number of important aspects of this problem remained to be investigated. The problem of time-variant stress and strength distributions needed further treatment. The effects of various factors, which are themselves distributions, on the distributions of the failure-governing stress and strength had yet to be fully explored. The development of a formal engineering design methodology for designing mechanical components had yet to be developed. Finally, much of the work in mechanical reliability theory suffered from a lack of statistically adequate data, due to a lack of test results on a large number of identical mechanical components.

The purpose of the current investigation is to fill in the gaps in the above-mentioned areas, with the following specific objectives:

1. Develop a formal engineering methodology for designing into mechanical components, subjected to combined-stress fatigue which involves time-dependent strength distributions, specified reliabilities.

2. Explore the methods of functions of random variables as applied to structural reliability.

3. Explore the methods available for determining failure-governing stress and strength distributions and develop new ones.

4. Explore the methods available for determining reliability once the failure-governing stress and strength distributions are known and develop new ones.

5. Develop and fabricate fatigue testing machines for reliability research, so that the explored and developed methodologies described above can be demonstrated.

6. Pursue a test program with a statistically significant number of test specimens to obtain data from which these methodologies can be demonstrated.

This first year's research includes literature research, theoretical research, design, development and fabrication of research equipment, experimental research, and computer programming efforts.

The scope of the research effort has been broad and comprehensive, and it has been intended that this report will discuss all aspects of the problem of designing mechanical components which are subjected to combined-stress fatigue by the design-by-reliability methodology. Included are all aspects of estimating distributions, estimating stress and strength factors, computing reliabilities by various methods, and experimental test procedures. This report should serve as a guide to the implementation of the design-by-reliability methodology and as a basis for future research.

The breadth of coverage has resulted in many areas being uncovered where more research in depth is required. These are mentioned in the body of the report and are also summarized in the overall Conclusions and Recommendations section.

The design-by-reliability methodology described in this report is a significantly new approach to the problem of design. This approach will enable the engineer and designer to proceed in a rigorous and scientific manner, according to a clearly defined method, to design a specified reliability into a mechanical component subjected to combined-stress fatigue. This is an important step forward in the science of designing a specified reliability into a mechanical or electro-mechanical components and from there into products and systems.

The first-year effort of the research on which this report is based dealt exclusively with the time-dependent strength distribution and the time-independent stress distribution. The time-dependent stress distribution case is intended to be the subject of future research.

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SECTION 1

DESIGN-BY-RELIABILITY METHODOLOGY FOR PARTS SUBJECTED TO COMBINED-STRESS FATIGUE LOADING

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CHAPTER 1.1

INTRODUCTION TO A SCIENTIFIC MECHANICAL COMPONENT DESIGN METHODOLOGY

Designing a specified reliability into a component requires that both its failure - governing stress and its failure - governing strength be treated as distributions. Moreover, the nominal failure - governing stress and strength must ordinarily be modified by a number of factors and conditions to bring them to the in-service failure - governing stress and strength.

Figure 1.1 (a) shows how this is done in practice. The nominal stress and the nominal strength are modified by various factors until the final distributions of failure - governing stress and failure - governing strength are determined. These distributions are shown in Fig. 1.1 (b).

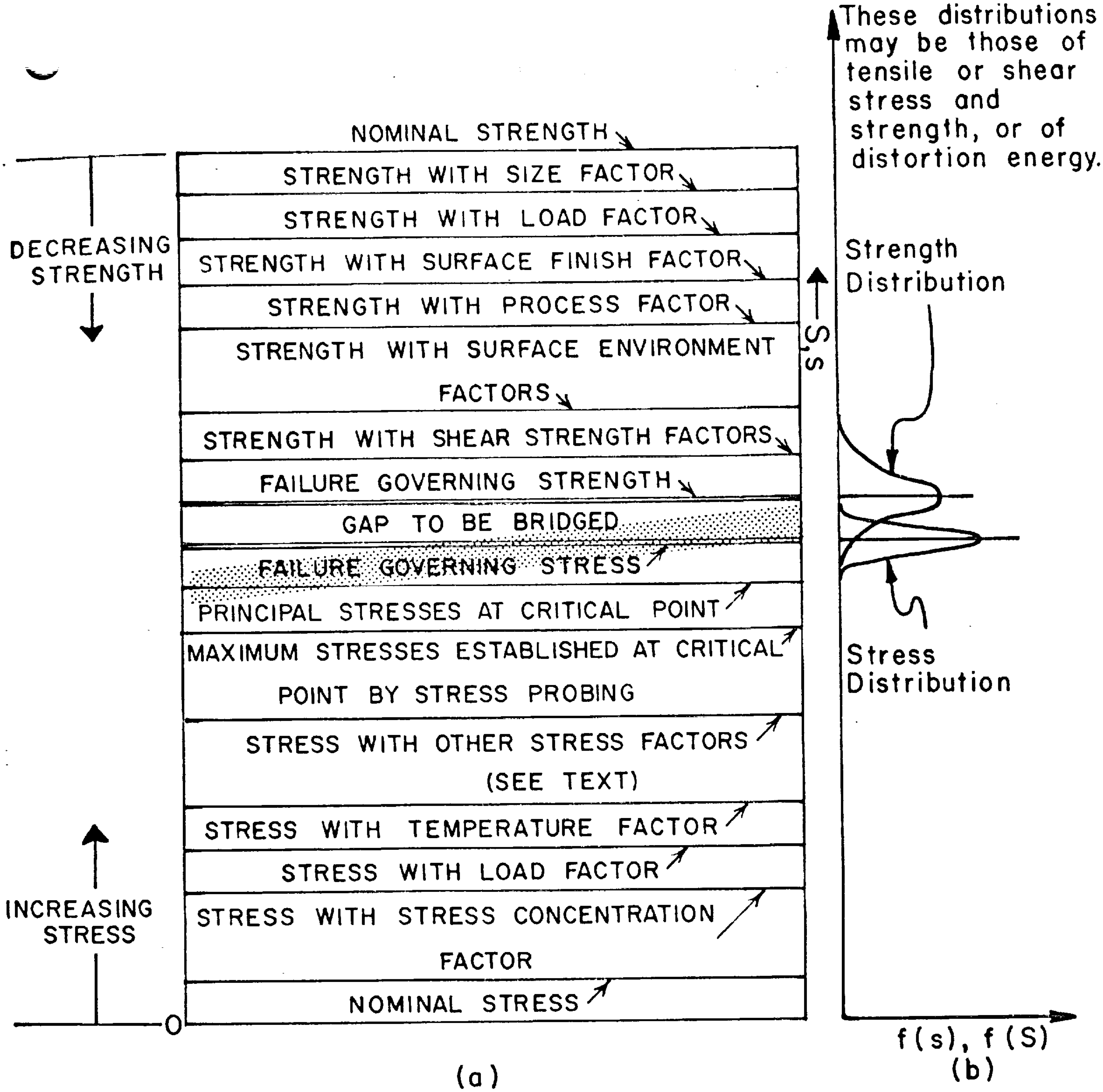
When the distributions of stress and strength have been determined, the reliability can be found as a measure of the interference of the stress and strength distributions. This will be discussed later in this Section, and further in Section 5 of this report.

Designing mechanical components by reliability is a relatively new endeavor and many engineers are not familiar with it. The main purpose of this Section is to introduce the engineer to the design by reliability methodology. A general engineering design methodology will first be presented. Then an application will be given in the form of an example problem. This problem will be presented from two viewpoints: a) first the problem will be worked by conventional engineering design methods, and b) then it will be worked by design by reliability methods. An example of designing a specified reliability directly into a shaft will be given. The parallel treatment of the same problem by both conventional and reliability methods should provide the engineer with an understanding of the design by reliability approach.

Also included in this Section is a discussion of the theories of material failure which are significant for determining the failure - governing stress and strength for parts subjected to combined-stress fatigue loading.

Next are included a number of other examples which illustrate briefly but completely how the design by reliability methodology can be extended to various loading and stress conditions, including both a finite-life and an infinite-life design.

First to be discussed will be a Scientific Mechanical Component Design Methodology.



(a). Stress increase and strength decrease resulting from the application of the respective stress and strength factors.

(b). Failure governing stress and strength distribution.

FIGURE 1.1 DETERMINATION OF THE FAILURE GOVERNING STRESS AND STRENGTH DISTRIBUTIONS.

CHAPTER 1.2

A SCIENTIFIC MECHANICAL COMPONENT DESIGN METHODOLOGY

The recommended scientific method of tackling and solving any problem, including that of mechanical design, consists of the following five basic steps (1)*. Some of the steps are broken into further subdivisions for elaboration.

1. State the problem.
2. Understand the problem.
3. Devise a plan of attack.
4. Carry out the plan.
5. Examine the solution -- Look back.

In mechanical design, the details of these steps are the following:

1. <u>State the problem.</u> - In clear and unambiguous terms verbally describe the problem, stating the objective, the given quantities and conditions, the constraints if any, and the available technical data, and provide the pertinent engineering drawings or sketches.

2. <u>Understand the problem.</u> - The problem may be understood best by drawing a complete sketch of the component and answering the questions: (a) what is the unknown? (b) what are the data? (c) what are the conditions and constraints? (d) what are the other factors?

3. <u>Devise a plan of attack.</u> - The best and most methodical plan of attack for designing a mechanical component, where the fiber stress governs its failure, involves the following three steps:

   3.1 Determine the failure governing stress.
   3.2 Determine the failure governing strength.
   3.3 Bridge the gap between the failure governing stress and strength.

4. <u>Carry out the plan.</u> - For the design of a mechanical component, carrying out the plan consists of successfully completing the three steps given previously. The details are the following:

   4.1 Determine the failure governing stress as follows:

      4.1.1 Perform stress probing
      4.1.2 Calculate the nominal stresses
      4.1.3 Determine the maximum value of each stress component involved.

This is accomplished by applying appropriate stress factors to the nominal stresses. These factors include, but are not necessarily limited to, the following:

---

* Number in parenthesis refer to references at the end of this Section.

1. Type of loading
2. Stress concentration
3. Manufacturing residual stress
4. Heat treatment
5. Assembly stress
6. Wear
7. Corrosion
8. Erosion
9. Cavitation
10. Temperature
11. Time

Values of these stress factors and procedures for applying them may be found in such references as (7), (8), (9), (10), (11), (12), and (13).

4.1.4 Determine the principal stresses

This step may or may not be required, depending on the convenient form of use of the failure governing stress criterion applicable to the particular problem at hand. References (2), (3), (4), (5), or (6) can be used.

4.1.5 Synthesize the stresses into the failure governing stress

It is very important that the correct failure governing stress is determined. The more commonly used failure governing stress criteria are the following:

1. Maximum normal or direct stress
2. Maximum shear stress
3. Maximum distortion energy
4. Combination of mean and alternating stresses with fatigue criteria

It is imperative that the criterion most applicable to the particular problem at hand be used. In the case of a shaft, for example, the maximum distortion energy criterion may be used to calculate the failure governing combination of the mean and alternating stresses. These stresses may then be combined on a modified Goodman diagram. The point in the shaft where the failure governing stress combination is closest to the failure governing strength line would be the point where failure is most likely to occur.

The problem of determining which failure governing stress theory applies in the case of combined-stress fatigue loading will be discussed in Chapter 3 of this Section.

4.2 Determine the failure governing strength

A major problem in design is to determine the strength which if exceeded leads to failure. For this, the strength criteria best associated with the type of failure involved should be selected.

4.2.1 Establish the applicable failure governing strength criterion.

This step is discussed in detail in Chapter 3 of this section.

4.2.2 Determine the nominal strengths

4.2.3 Synthesize the strengths into the failure governing strength.

Nominal strength values arrived at by tests on specimens of different geometry, size, surface, treatment and other conditions, should be converted to the actual strength that will be exhibited by the component being designed in its actual application and operation environment. To convert nominal strengths to actual strengths, strength factors are used. These factors may include, but are not necessarily limited to those discussed in Section 4.2.3 as well as, surface environment, surface treatment, notch sensitivity, size, surface finish, manufacturing processes, fatigue and creep.

The next step is to apply the modified nominal strengths to the failure governing strength criterion. This completes the steps to arrive at the failure governing strength.

4.3 Bridge the gap

The gap of concern is that between the failure governing strength and the failure governing stress. Such a gap is the one indicated by the shaded gap in Fig. 1.1 (a). For fatigue, it is the shaded gap in Fig. 1.5 or the distance from $\bar{s}_f$ to $\bar{S}_f$.

Three approaches may be used to bridge this gap:

1. Safety factor
2. Safety margin
3. Reliability

The safety factor, S. F., is usually defined as

$$S.F. = \frac{\bar{S}_f}{\bar{s}_f} \qquad (1.2.1)$$

It is the ratio of the failure governing strength mean, $\bar{S}_f$, to the failure governing stress mean, $\bar{s}_f$.

The safety margin, S. M., is usually defined either as

$$S.M. = S.F. - 1 \qquad (1.2.2)$$

or as

$$S.M. = \frac{\text{Mean Strength-Maximum Stress}}{\text{Strength Standard Deviation}} = \frac{\bar{S} - s_{max}}{\sigma_S} \qquad (1.2.3)$$

where $s_{max} = \bar{s} + 4.5\, \sigma_s$ and the factor 4.5 is chosen arbitrarily.

The reliability approach will be discussed later.

5. Examine the Solution - Look Back. - This step consists of answering the following questions after examining the solution:

a. Can you check the result? Can you check the argument?
b. Can you derive the result differently? Can you see it at a glance?
c. Can you use the result, or the method, for some other problem?

The steps of the formal design methodology will be illustrated next by an example problem. The problem, when carried through by both the conventional design methods and the design by reliability methods, will serve to illustrate the difference in the two approaches.

Conventional Design Example

1. Statement of the Problem

A solid, round, rotating shaft is to be loaded by a bending moment of 6,000 lb.-in. in one plane, of 10,000 lb.-in. in a plane 45 degrees clockwise to the first and of 8,000 lb.-in. in a plane 90 degrees clockwise to the first. All of these three planes contain the axis of rotation of the shaft and all moments are of the same sign. An axial, compressive load of 5,000 pounds and a torque of 15,000 lb.-in. are also to act on this shaft.

The material to be used is cold drawn ductile steel having a Su = 100,000 psi and a Sy = 72,000 psi. The critical point of the shaft has a theoretical stress concentration factor of 1.5 in bending. The fillet radius at this point is 1/4 in. Consider no other stress concentration factors.

Using a safety factor of 2.0 and a 95 per cent reliability at 10^6 cycles, find the appropriate shaft diameter based on the Von Mises-Hencky-Goodman strength criterion.

2. Understanding the Problem

A sketch of the shaft and its loading is given in Fig. 1.2.

What is the unknown?

The shaft diameter, d , at the critical point, i.e., at the point where the shaft is most likely to fail.

What are the data?

1. Solid, round, rotating shaft.
2. Loads
   a. $M_1 = 6,000$ lb.-in.
   b. $M_2 = 10,000$ lb.-in. in a plane 45^o clockwise to the first.

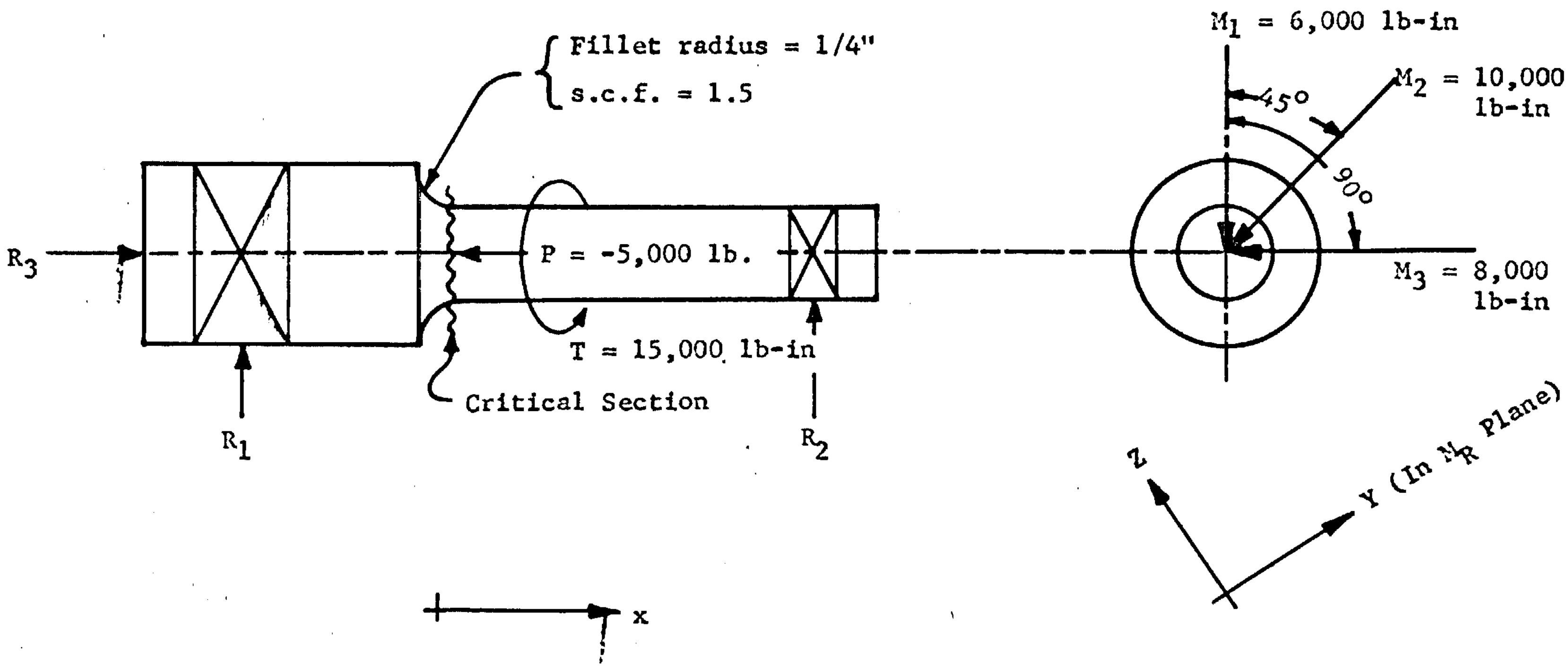


FIGURE 1.2. ROTATING CIRCULAR SOLID SHAFT WITH LOADS ACTING ON IT.

$M_3 = 8{,}000$ lb.-in. in a plane 90° to the first.

b. Axial load: $P = -5{,}000$ lb.
c. Torque: $T = 15{,}000$ lb.-in.

3. Material: Cold drawn ductile steel with

$Su = 100{,}000$ psi.
$Sy = 72{,}000$ psi.

4. Stress concentration factors: At the critical point s.c.f = 1.5 in bending; consider none other.

What are the conditions?

1. Safety factor = 2.
2. Desired reliability = 95% at 10^6 cycles.
3. The Von Mises-Hencky-Goodman strength criterion applies.

What are the other factors?

1. The information is sufficient.
2. All conditions can be met except perhaps that of simultaneously designing for a safety factor of 2 and a reliability of 95% at 10^6 cycles. This aspect will be discussed later.

3. Devising a Plan of Attack

For such mechanical components the plan of attack has already been presented in Chapter 2, Item 3, and consists of

3.1 determining the failure governing stress,
3.2 determining the failure governing strength, and
3.3 bridging the gap

4. Carrying out the Plan

4.1 Determining the failure governing stress

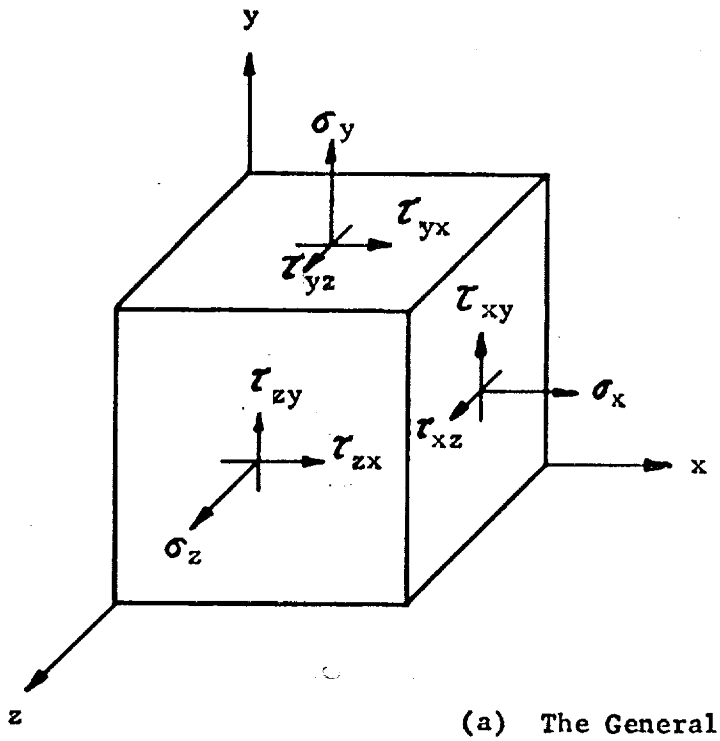
Perform stress probing

The way the given problem is stated, this step is not required, because the critical point on this shaft where a failure is most likely to occur is given to be the fillet designated in Fig. 1.2.

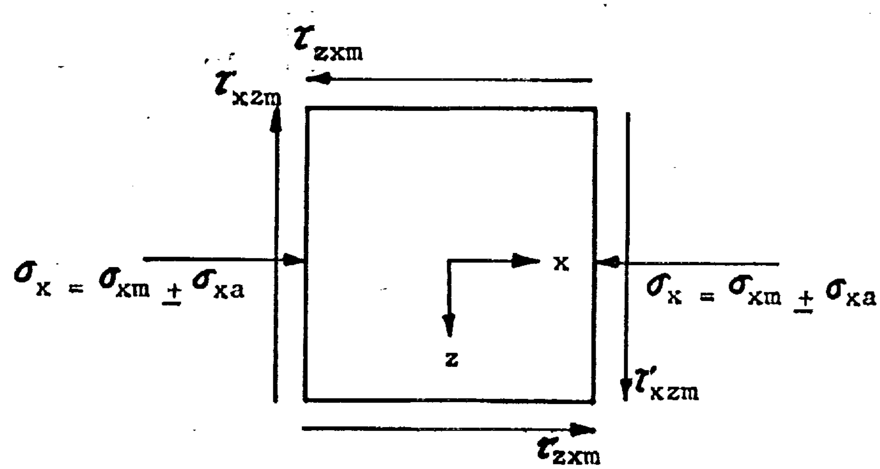
Calculate the nominal stresses

First, we have to determine what stresses are acting. The general stresses that may act on an element of the shaft at the fillet are shown in Fig. 1.3.

For this problem, however, if the resultant moment's plane is taken to be the ~~x~~ Z-y plane, and the shaft's axis of rotation is taken to be the x-axis, then



(a) The General Stress Case.



(b) The Specific Case.

FIGURE 1.3. STRESSES ACTING ON AN ELEMENT IN A ROTATING SHAFT.

$$\sigma_y = \sigma_z = 0 \tag{1.2.4a}$$

$$\tau_{xy} = \tau_{yz} = 0 \tag{1.2.4b}$$

and the remaining stresses are σ_x and τ_{zx} .

The shaft is subjected to fatigue; consequently, the σ_x stress has two components: (1) the constant stress component due to the constant compressive load P = -5,000 lb., and (2) the variable or alternating component due to the resultant bending moment acting on the rotating shaft. Therefore

$$\sigma_x = \sigma_{xm} + \sigma_{xa} \tag{1.2.5}$$

From equilibrium and combined-stress fatigue conditions

$$\tau_{xz} = \tau_{zx} = \tau_{xzm} + \tau_{xza} \tag{1.2.6}$$

As the shaft is subjected to a constant torque, $\tau_{xza} = 0$, and only τ_{xzm} is left. Therefore

$$\tau_{xz} = \tau_{zx} = \tau_{xzm} \tag{1.2.7}$$

The nominal values of these stresses are calculated as follows:

$$\sigma_{xm} = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(-5,000)}{\pi d^2}$$

$$\sigma_{xm} = -6,360/d^2$$

$$\sigma_{xa} = \frac{M_R c}{I} = \frac{M_R d/2}{\pi d^4/64} = \frac{32\ M_R}{\pi d^3} \tag{1.2.8}$$

where M_R = 20,000 lb.-in., from Fig. 1.4. Hence,

$$\sigma_{xa} = \frac{32 \times 20,000}{\pi d^3}$$

$$\sigma_{xa} = 204,000/d^3 \tag{1.2.9}$$

$$\tau_{xzm} = \frac{Tc}{J} = \frac{T\ d/2}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

$$\tau_{xzm} = \frac{16 \times 15,000}{\pi d^3}$$

$$\tau_{xzm} = 76,400/d^3 \tag{1.2.10}$$

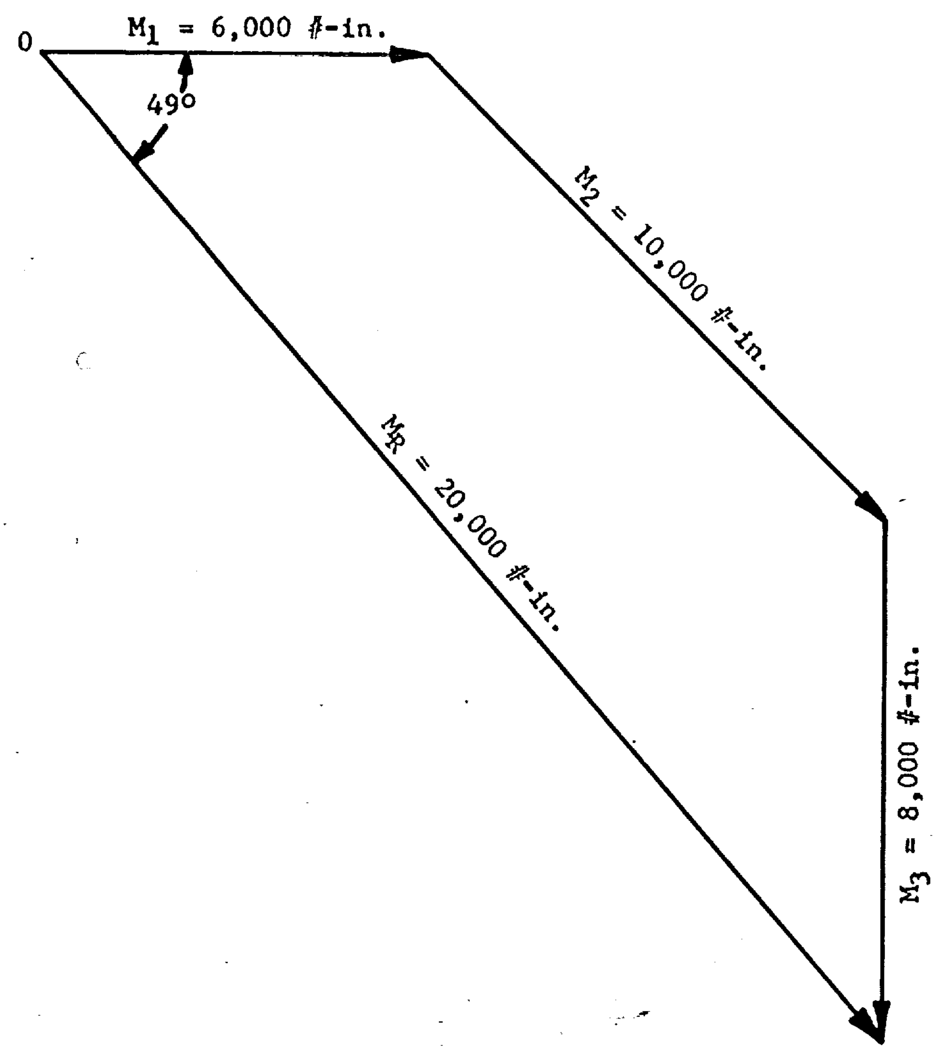


FIGURE 1.4. GRAPHICAL DETERMINATION OF THE RESULTANT MOMENT ACTING ON THE SHAFT.

Determine the maximum value of each stress component involved

In this problem, σ_{xm} and τ_{xzm} are the maxima already, but not σ_{xa} because a stress concentration exists in the fillet. Then

$$\text{Maximum } \sigma_{xa} = \sigma_{xa} \times K_f \tag{1.2.11}$$

where

K_f = fatigue stress concentration factor

and is given by

$K_f = 1 + q(K_t - 1)$. See Shigley (7, p. 170, Eq. 5-18)

$q = 0.9$. See Shigley (7, p. 171, Fig. 5-27)

K_t = theoretical stress concentration factor = 1.5, as given in the statement of the problem.

Therefore

$$K_f = 1 + 0.9\ (1.5-1) \tag{1.2.12}$$

$$K_f = 1.45$$

and $\sigma_{xa} = \dfrac{204,000}{d^3} \times 1.45$

or $$\sigma_{xa} = 295,000/d^3 \tag{1.2.13}$$

for the maximum value of that strèss component.

Determine the principal stresses

The failure governing stress can be expressed in terms of σ_x and τ_{xz} , in this problem, hence the principal stresses need not be calculated.

Synthesize the stresses into the failure governing stresses

As stated in the problem, the maximum distortion energy or the Von Mises-Hencky criterion governs because the shaft is of ductile material subjected to fatigue. In Chapter 3 of this Section, the appropriate theories of failure for combined-stress fatigue will be discussed more thoroughly.

For the ordinary element, the mean and alternating components of the Von Mises-Hencky failure governing stresses are given by

$$s_m = \sqrt{\sigma_{xm}^2 - \sigma_{xm}\,\sigma_{zm} + \sigma_{zm}^2 + 3\,\tau_{xzm}^2} \tag{1.2.14}$$

$$s_a = \sqrt{\sigma_{xa}^2 - \sigma_{xa}\,\sigma_{za} + \sigma_{za}^2 + 3\tau_{xza}^2} \qquad (1.2.15)$$

from Shigley (7, p. 138, Eqs. 5-29), which reduce to the following for this problem:

$$s_m = \sqrt{\sigma_{xm}^2 + 3\tau_{xzm}^2} \qquad (1.2.16)$$

$$s_a = \sqrt{\sigma_{xa}^2} = \sigma_{xa} \qquad (1.2.17)$$

The maximum stresses calculated previously can now be substituted into the equations for failure governing stresses as follows:

$$s_m = \sqrt{(-6{,}360/d^2)^2 + 3(76{,}400/d^3)^2}$$

and

$$s_a = 295{,}000/d^3 \qquad (1.2.18)$$

If d was known, then s_m and s_a could have been calculated and plotted on the Modified Goodman Diagram to see their relative position with respect to the failure governing strength. This cannot be accomplished as yet; however, the locus of the failure governing stress on the Modified Goodman Diagram can be determined. The locus will be a line with the slope s_a/s_m or

$$\text{Slope} = \frac{s_a}{s_m} = \frac{295{,}000/d^3}{\sqrt{(-6{,}360/d^2)^2 + 3(76{,}400/d^3)^2}} \qquad (1.2.19)$$

The slope cannot as yet be calculated, unless this expression is simplified. An inspection of the denominator reveals the first term is very small numerically in comparison with the second term, hence the first term may be dropped as a first approximation. The validity of this assumption should be checked back, however, after the final result is obtained. Then

$$\frac{s_a}{s_m} = \frac{295{,}000/d^3}{\sqrt{3(76{,}400/d^3)^2}} = \frac{295{,}000/d^3}{\sqrt{3(76{,}400/d^3)^2}} = \frac{295{,}000}{132{,}000}$$

Therefore

Slope = 2.24.

This slope line has been drawn in Fig. 1.5. This is as far as the failure governing stress determination aspect of the problem can be carried out. It can be determined, however, once the gap between it and the failure governing strength is bridged. We shall determine the latter next.

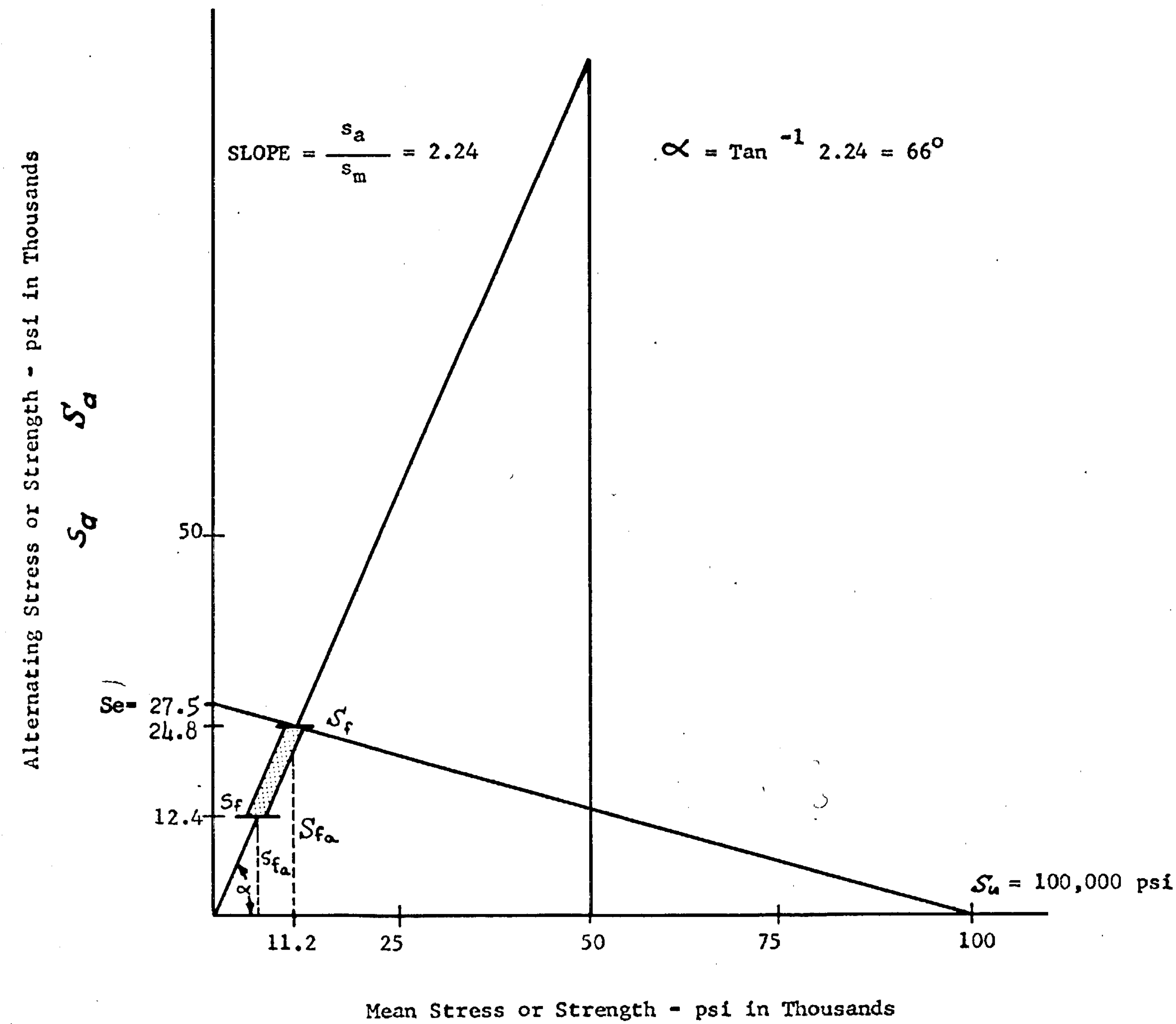


FIGURE 1.5. MODIFIED GOODMAN DIAGRAM

4.2 Determining the failure governing strength

Establish the applicable failure governing strength criterion

As stated in the problem, the Von Mises-Hencky-Goodman fatigue strength criterion applies to this problem. The applicable strength criterion is the line drawn in Fig. 1.5 from S_e to S_u .

Determine the nominal strength

The nominal endurance strength, S_e' , as obtained from the standard fatigue is

$S_e' = 0.50\ S_u$* from Shigley (7, Eqs. 5-11, p. 162) because the shaft will be made of steel with an ultimate strength of $S_u = 100{,}000$ psi which is less than 200,000 psi. If $S_u \geq 200{,}000$ psi, then $S_e' = 100{,}000$ psi* is used for steels. Therefore, for this problem

$$S_e' = 0.50 \times 100{,}000 = 50{,}000 \text{ psi.}$$

The nominal static strength is the ultimate tensile strength, S_u , and the value given for this in the problem is $S_u = 100{,}000$ psi.

Synthesize the strengths into the failure governing strength

a. To obtain the actual endurance strength, the nominal endurance strength should be corrected using the applicable strength factors, as illustrated in Figure 1.1(a).

Then

$$S_e = S_e'\ k_a\ k_b\ k_c\ k_d\ k_f \tag{1.2.20}$$

where

S_e' = endurance limit of the rotating-beam specimen, in psi. This is not the strength that will be exhibited by the shaft in its actual geometry, and application and operation environments. Hence it is only the nominal strength and will have to be modified to apply it to this problem.

k_a = surface finish factor

k_b = size factor

k_c = reliability factor

k_d = temperature factor

*This relationship is one of a rule-of-thumb. Both the 0.50 coefficient and the value for S_u need to be studied further from a distributional point of view so that the true distribution of S_e' can be found.

k_f = miscellaneous-effects factor

Shigley (7) recommends the incorporation of a reliability factor in this manner for determining the actual endurance strength. It must be pointed out that this approach is not correct, and that this research points out the correct approach. The factor will be retained at this point however, so that the conventional method can be illustrated.

It should also be noted that k_e, the stress concentration factor, is not included here because it should preferably not be applied to strength. This is a departure from the methods of conventional design as given in Shigley (7, p. 166). There may be a multiplicity of stress concentration factors involved, each one having different magnitude, with a net result that may not at all be equivalent to a factor which when applied to the nominal strength will duly take into account all s.c.f.'s and the notch sensitivity. As it may be seen, in this problem only σ_{xa} experiences a stress concentration factor. Furthermore, it does not have the same effect as either multiplying the failure governing stress or dividing the failure governing strength by the s.c.f.

These strength factors may be determined as follows:

Surface factor, k_a = 0.74 (7, Figs. 5-26, p. 167)
Size factor, k_b = 0.85 (7, p. 168)
Reliability factor, k_c = 1-(0.08 x 1.6) = 1-0.128 = 0.872 (7, Table 5-2, p. 169)
Temperature factor, k_d = 1, as no temperature effect is indicated in the problem.
Miscellaneous-effects factor, k_f = 1, as none other is indicated in the problem.

Consequently, the failure governing endurance strength is

$$S_e = 50{,}000 \times 0.74 \times 0.85 \times 0.872 \times 1 \times 1$$

or $$S_e = 27{,}500 \text{ psi.}$$

b. The failure governing strength in static loading is S_u because fracture is considered failure of the shaft, or S_u = 100,000 psi. If in a particular design application, yielding is selected as the static strength failure criterion, then S_y should be used for this strength, instead of S_u, along the abscissa.

c. The failure governing strength in fatigue is conservatively taken to be the line joining S_e (at 10^6 cycles for this problem) and S_u . The closer the (s_a,s_m) point is to this line, the greater is the chance of failure. This failure chance approaches 50% when the point is on the strength line, and approaches 100% as the point moves away from the strength line and substantially to the right of it.

4.3 Bridging the Gap

In this problem, a safety factor of 2 is specified, hence,

$$\frac{S_f}{s_f} = 2$$

also

$$\frac{S_{fa}}{s_{fa}} = 2 \text{ and } \frac{S_{fm}}{s_{fm}} = 2$$

from similar triangles in Fig. 1.5.

Therefore, using the alternating components,

$$\frac{S_{fa}}{s_{fa}} = 2 = \frac{24,800}{295,000/d^3} \tag{1.2.21}$$

Consequently,

$$d^3 = 23.9$$

and $$d = 2.87 \text{ inches}$$

say $$d = 2\text{-}7/8 \text{ inches}$$

This is the solution to the example problem.

5. Examining the Solution

a. Checking the solution reveals that it is correct based on the safety factor concept.

b. An assumption was made in determining the slope s_a/s_m (Eq. 1.2.19). The validity of this assumption should be checked. The first term in the expression for s_m was considered to be negligible with respect to the second term. Actually,

$$\frac{[-6,360/(2.875)^2]^2}{3\,[76,400/(2.875^3)]^2} \times 100 = \frac{5.95 \times 10^5 \times 10^2}{3.12 \times 10^9} \cong 0.02\%$$

or indeed negligible; thus validating the assumption.

c. Alternate

Also, the fact that $\frac{S_{fm}}{s_{fm}} = 2$ can be used. The same result as before would be obtained, of course.

d. The answer is, therefore, correct within the approach used.

e. The answer is not quite correct on the basis of designed-in reliability, however, as it will be shown next.

Design by Reliability

Every designer essentially attempts to design for minimum or essentially no failures, hence he intuitively thinks of reliability, or the probability that the product will not fail. However, using safety factors, or safety margins, gives relatively little quantitative indication of the reliabilities involved. Fortunately, methods do exist now which enable the designer to calculate and thus predict component and product reliabilities.

The design method to be discussed here does not accept the theory that the failure governing stress and the failure governing strength are single-valued quantities. Rather, it accepts the theory that variabilities in the loads, moments, torques, geometries, physical properties, manufacturing processes and procedures, and environmental factors which affect the stresses in identical components in a fleet of products, say in the same shaft of a fleet of trucks or jet aircraft engines, will result in a failure governing stress distribution, instead of a single value of stress. In other words, if we had strain gages on the particular shaft fillet of every truck or jet engine in this fleet performing the same task, and determined the stress history for every identical trip or flight, picked up the maxima of the stresses in each flight, converted them to failure governing stresses, and plotted how many times a particular small range of these stress values occurred, a distribution like that shown in Fig. 1.6 would be obtained. As actually all the shafts for the fleet of such trucks or jet airplanes and not a single shaft are designed, for the shafts in these fleets to survive, they should be designed for the failure governing stresses to be experienced by the identical shaft in the whole fleet. Hence the design should be based on this distribution of the failure governing stresses in the shaft and not on some single value: maximum, mean or other.

Similarly, the strength exhibited by these shafts will not be single-valued in reality. Variabilities in the heat of metal the shafts are made of, the manufacturing process, the various treatments the shaft is subjected to, the assembly process, and the environment the shafts will see in the fleet will result in a distribution of the failure governing strengths of identical shafts. In other words, some shafts will fail at a substantially lower level of imposed stress because they are weaker, and some will fail only when a substantially high level of stress is imposed on them because they are stronger than the average strength shaft.

Consequently, for a component in a fleet of products, the designer has to consider and deal with the joint stress and strength distributions shown in Fig. 1.7a. The shaded area gives numerically the probability of failure of such a component experiencing the stress distribution given and exhibiting the strength distribution given in Fig. 1.7a. A method for calculating the reliability of a component having normal stress and strength distributions will be presented next.

Determination of Component Reliability When the Failure Governing Stress and Strength Distributions are Normal

If the density functions $f(s)$ and $f(S)$, representing the stress and strength distributions, respectively, are Gaussian or normal, then they may be expressed as

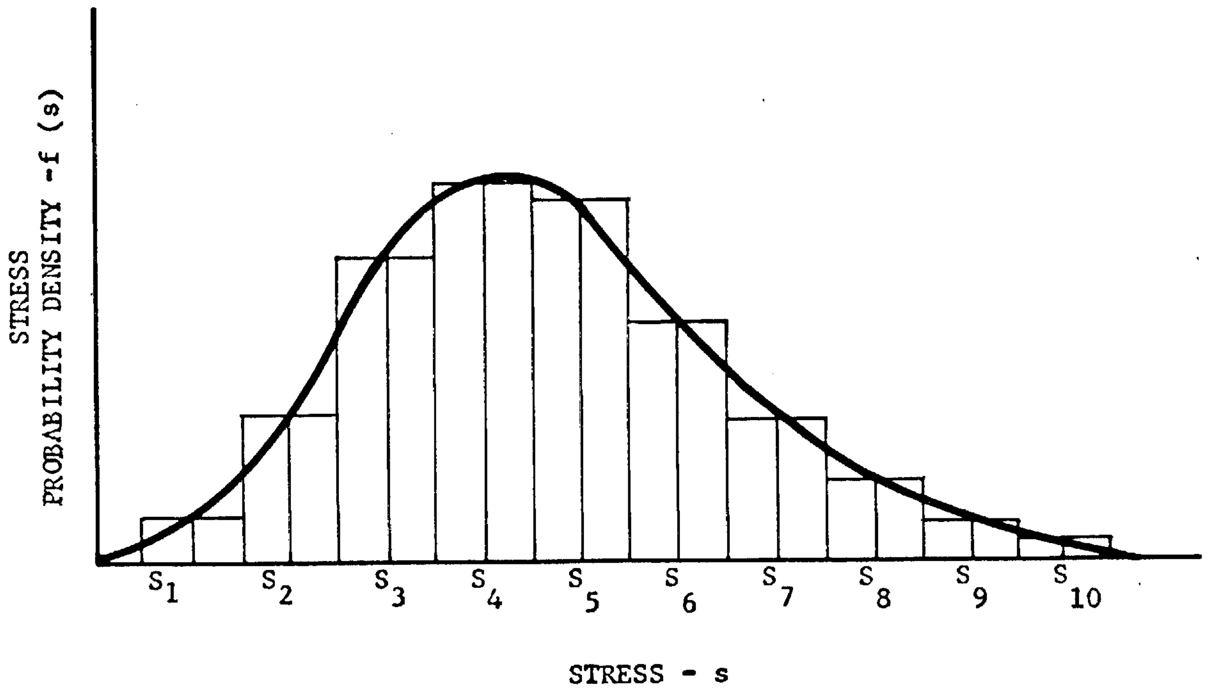


FIGURE 1.6. STRESS DISTRIBUTION

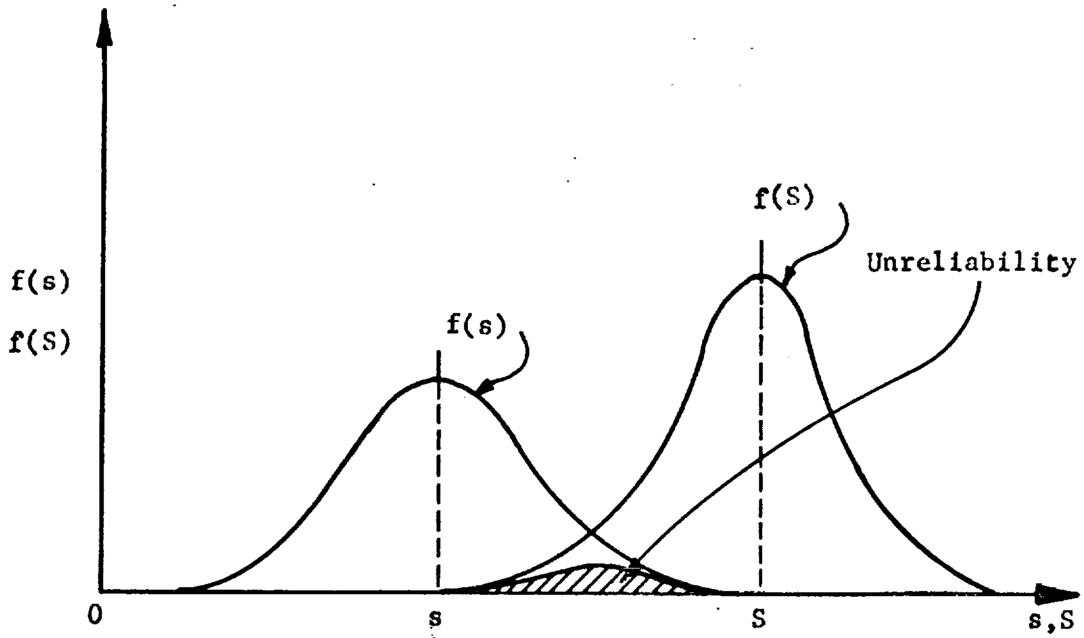


FIGURE 1.7(a). STRESS AND STRENGTH DISTRIBUTIONS.

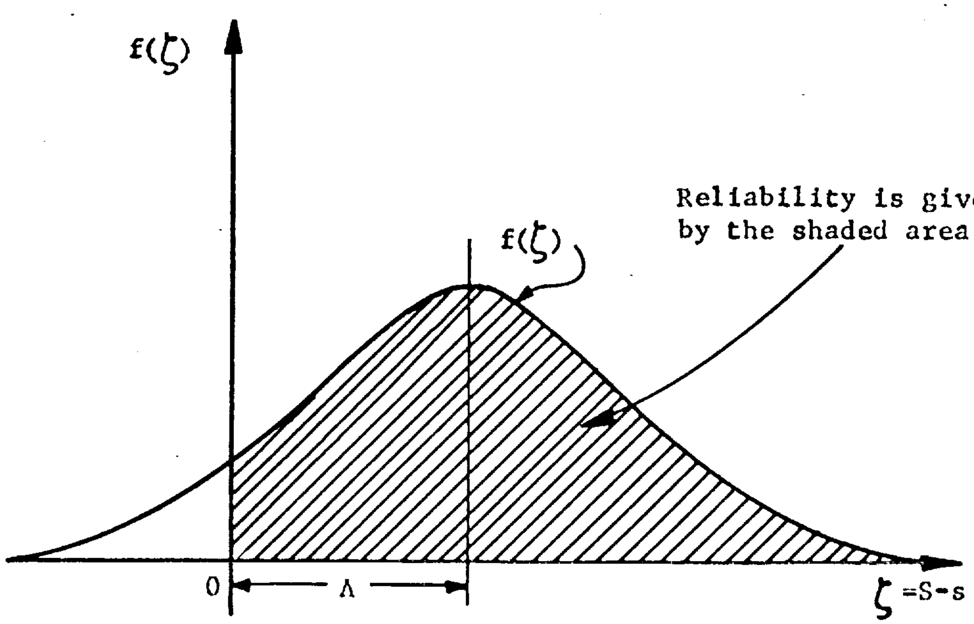


FIGURE 1.7(b). DISTRIBUTION OF DIFFERENCE FUNCTION OF STRENGTH AND STRESS, f (ζ).

$$f(s) = \frac{1}{\sigma_s\sqrt{2\pi}} \; e^{-\frac{1}{2}\left[\frac{s-\bar{s}}{\sigma_s}\right]^2} \tag{1.2.22}$$

and

$$f(S) = \frac{1}{\sigma_S\sqrt{2\pi}} \; e^{-\frac{1}{2}\left[\frac{S-\bar{S}}{\sigma_S}\right]^2} \tag{1.2.23}$$

where

$\bar{s}$ = mean of the failure governing stress distribution

σ_s = standard deviation of this stress distribution

$\bar{S}$ = mean of the failure governing strength distribution

σ_S = standard deviation of this ~~stress~~ strength distribution

Reliability is given by all probabilities that strength is in excess of stress or that $S-s > 0$. Using the designation $\zeta = S-s$, reliability is given by all of the probabilities that $\zeta > 0$. $h(\zeta)$ is defined as the difference distribution of $f(S)$ and $f(s)$, and as $f(S)$ and $f(s)$ are normally distributed, then $h(\zeta)$ is normally distributed also (16, pp. 215-216) and is expressed by

$$h(\zeta) = \frac{1}{\sigma_\zeta\sqrt{2\pi}} \; e^{-\frac{1}{2}\left[\frac{\zeta-\bar{\zeta}}{\sigma_\zeta}\right]^2} \tag{1.2.24a}$$

where

$\bar{\zeta} = \bar{S} - \bar{s}$ = mean of the difference distribution (1.2.24b)

and

$\sigma_\zeta = \sqrt{\sigma_S^2 + \sigma_s^2}$ = standard deviation of the difference distribution

(1.2.24c)

Reliability would then be given by all probabilities of ζ being a positive value, hence

$$R = \frac{1}{\sigma_\zeta\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{1}{2}\left[\frac{\zeta-\bar{\zeta}}{\sigma_\zeta}\right]^2} d\zeta \tag{1.2.25}$$

The function $h(\zeta)$ and the value of R are shown in Fig. 1.7b.

The relationship between $h(\zeta)$ and the standardized normal distribution can be utilized to evaluate the above integral. The transformation relating ζ and the standardized variable t may be used which is

$$t = \frac{\zeta - \bar{\zeta}}{\sigma_\zeta} \qquad (1.2.26)$$

The new limits of the integrand are

for $\zeta = 0$, $t = \dfrac{0 - \bar{\zeta}}{\sigma_\zeta} = -\dfrac{\bar{\zeta}}{\sigma_\zeta}$

and for $\zeta = +\infty$, $t = \dfrac{+\infty - \bar{\zeta}}{\sigma_\zeta} = +\infty$

also $d\zeta = \sigma_\zeta \, dt$

If these conditions are substituted into the reliability equation, the following result is obtained:

$$R = \int_{-\frac{\bar{\zeta}}{\sigma_\zeta}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2} dt \qquad (1.2.27)$$

Consequently, the reliability of a component is given by the area under the standardized normal density function from the value of

$$t = -\frac{\bar{\zeta}}{\sigma_\zeta} \qquad \text{to} \quad t = +\infty$$

The value of this area may be obtained from the tables of areas under the standardized normal density function, available in many references (9, p. 589). Reliability as a function of t is plotted in Fig. 1.8.

Reliability Design Example

The values found in the problem at the beginning of this Section for the failure governing stress and strength will be used. These are

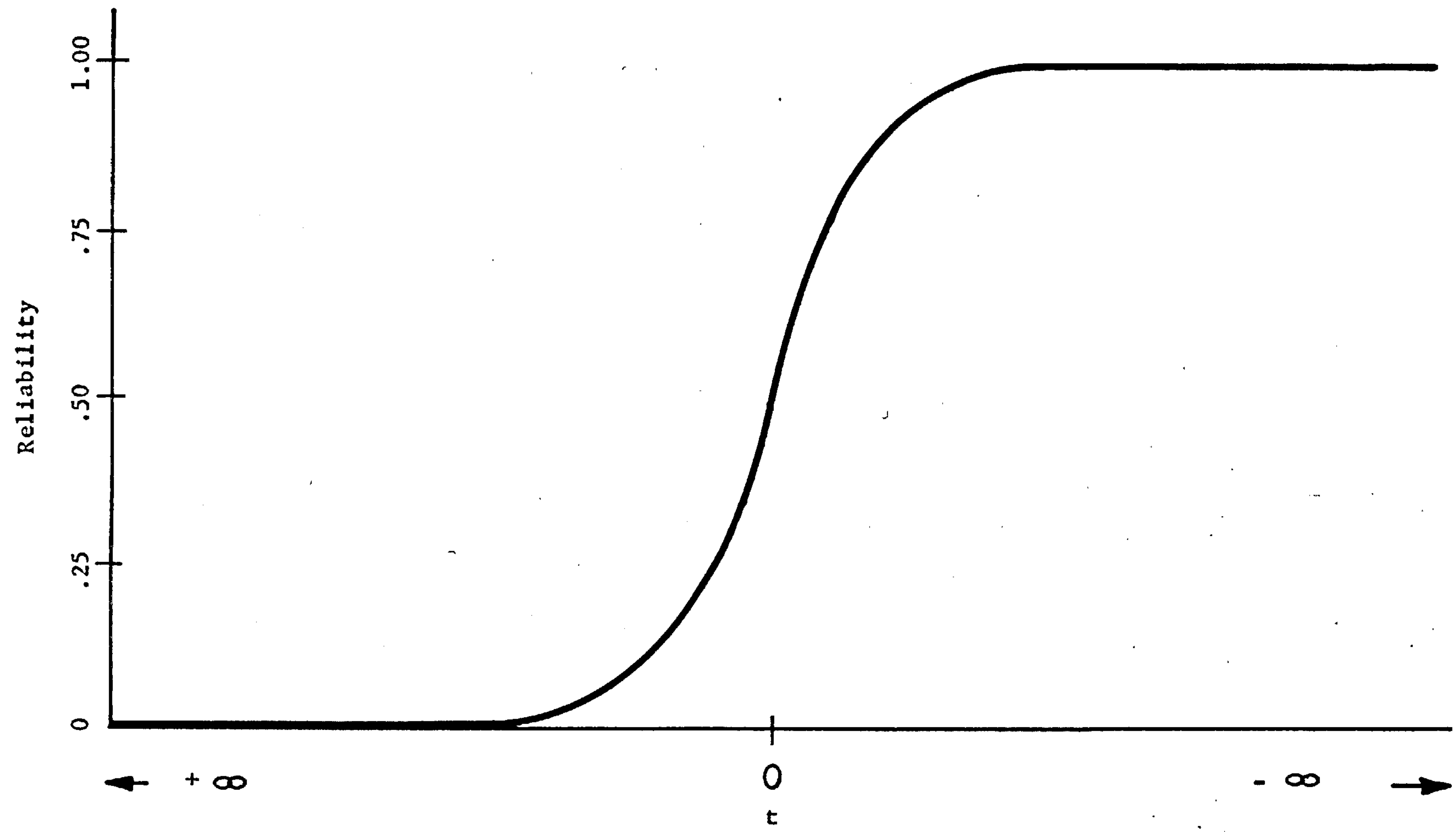


FIGURE 1.8. RELATIONSHIP BETWEEN RELIABILITY AND THE STANDARDIZED NORMAL VARIATE "t"

$$\overline{S}_f = \frac{24,800}{\sin \alpha} = 27,300 \text{ psi}$$

From Fig. 1.3, tan α = 2.24 and α = 66°

$$\overline{s}_f = \frac{\overline{S}_f}{2} = 13,650 \text{ psi}$$

As σ_s and σ_S are not known, for σ_s a realistic value of 3,000 psi and for σ_S the value of 8% of $\overline{S}_f$ or 2,200 psi recommended by Shigley (7, p. 169), will be used.* Then

$$\overline{\zeta} = \overline{S}_f - \overline{s}_f = 27,300 - 13,650 = 13,650 \text{ psi}$$

$$\sigma_\zeta = \sqrt{\sigma_S{}^2 + \sigma_s{}^2} = \sqrt{(2,200)^2 + (3,000)^2} = \sqrt{13.84 \times 10^6}$$

$$\sigma_\zeta = 3,700 \text{ psi}$$

$$t = -\frac{\overline{\zeta}}{\sigma_\zeta} = -\frac{13,650}{3,700} = -3.7$$

Then

$$R = \int_{-3.7}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

From normal function area tables, the area from t = -3.7 to t = + ∞ is 0.99989; therefore, the reliability is for practical purposes equal to 1. This compares with R = 0.95 to which the shaft was sized using Shigley's, or the conventional method (7, p. 169). Because of this foreseen discrepancy, the statement appearing at the end of "Example - Examining the Solution" was made. The reasons for this discrepancy are the following:

1. The reliability factor of conventional design is based on the distribution of strength beyond the knee of the S-N simple fatigue curve shown in Fig. 1.9, where the mean stress is zero and the alternating stress is one of complete reversal, hence the ordinate of Fig. 1.9. is S_a .

---

* This is a rule of thumb value. Another value which has been suggested is 4%, in "Symposium on Fatigue with Emphasis on Statistical Approach II," ASTM STP No. 137, June 24, 1952, p. 52. This standard deviation of the endurance limit is another area where much research needs to be done.

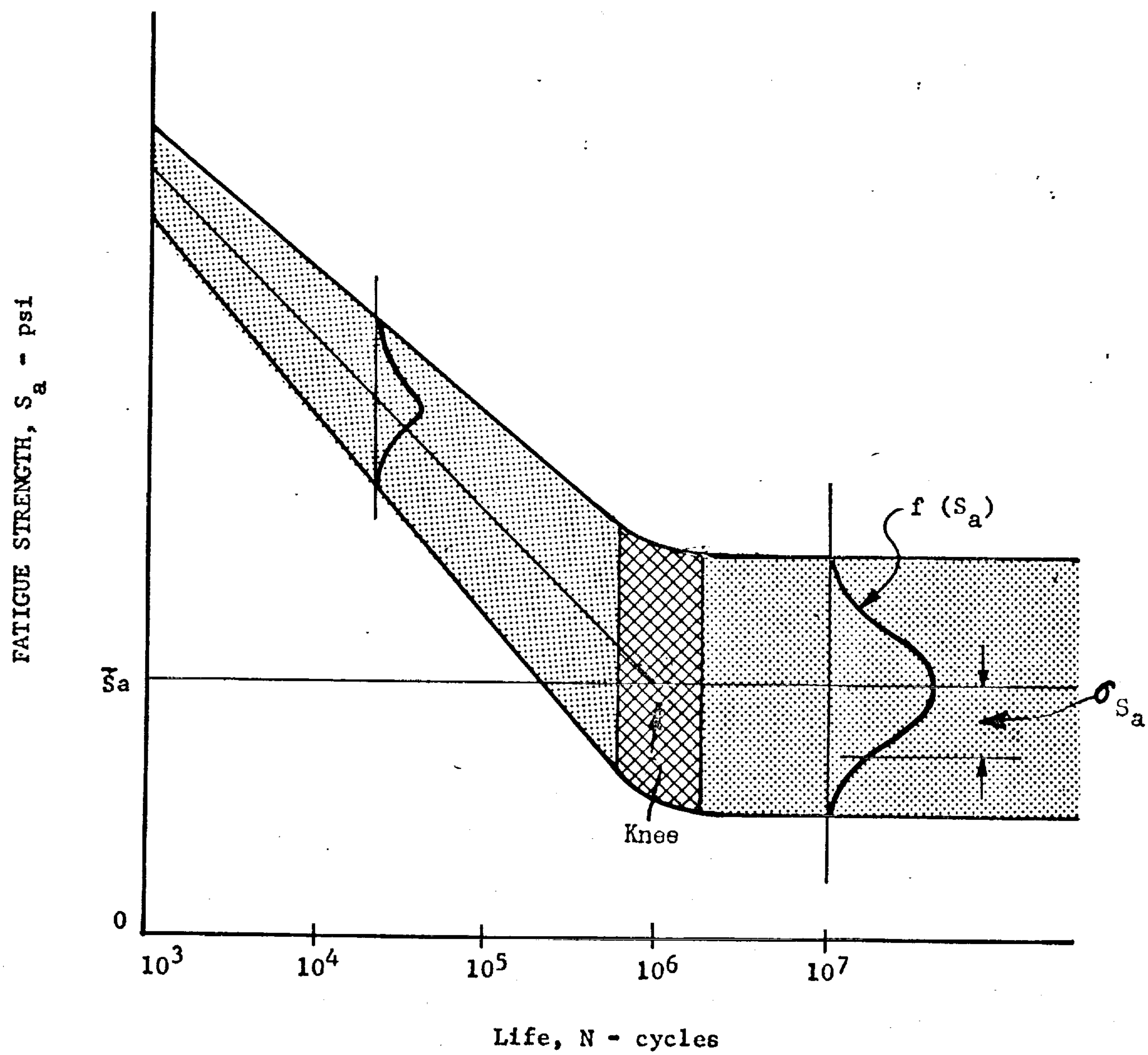


FIGURE 1.9. STRENGTH - CYCLE CURVE FOR SIMPLE FATIGUE.

In the example problem solved in the beginning of this chapter, there exist both mean and alternating stress components. Shigley's and conventional design fatigue factors should be used only for simple fatigue cases.

2. The conventional design reliability factor is based on the fatigue strength distribution of a standard, rotating-beam fatigue specimen, and the factors are applied to the mean of this distribution to convert it to the specific case of the problem. This application merely adjusts the mean strength and does not adjust the standard deviation which most probably is also affected by other factors, such as size, surface finish, temperature, etc. The value of 8% of the $\bar{S}_f$ used in the problem is not truly applicable in this case because it should be the standard deviation of the strength distribution with all the correction factors applied and not of the distribution for the standard rotating-beam specimen.

3. Conventional design methodology assumes that the failure governing stress is represented by a single value which would be its mean in this problem. In the illustrative problem, the failure governing stress was taken to be a normal distribution with a mean of 13,650 psi and a standard deviation of 3,000 psi. If the standard deviation is taken to be zero to correspond to the single-valued case, then the reliability would be given by

$$R = \int_{-\frac{\bar{\zeta}}{\sigma_\zeta}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

where $\sigma_\zeta = \sigma_S$ because $\sigma_s = 0$

or

$$R = \int_{-\frac{\bar{\zeta}}{\sigma_S}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

where $t = -\frac{\bar{\zeta}}{\sigma_S} = -\frac{13,650}{2,200} = -6.21$

Therefore, from tables of areas under the normal distribution

$$R = 0.9^9269 \cong 1$$

This compares with $R = 0.95$ used in the example problem in the beginning of this chapter, and is greater than $R = 0.99989$ obtained with a $\sigma_s = 3,000$ psi.

4. The true failure governing stress and strength distributions for the illustrative problem are those shown in Fig. 1.10. Here the assumption is made that the ratio of the alternating to the mean stresses in fatigue in this problem remains always constant. It is with this assumption that $f(s)$ is drawn as a two dimensional rather than a distribution of more than two dimensions. See reference (17, Fig. 3) for the multi-dimensional case. The two distributions in Fig. 1.10 are the ones that should be determined by the methodology presented in this chapter and then the reliability resulting from these distributions should be calculated to obtain the true component reliability.

5. It should be pointed out that these distributions have been taken to be normal. There are strong indications that these distributions may be non-normal. Among the non-normal distributions that are found to fit better are the log-normal and Weibull. The determination of reliability with such failure governing stress and strength distributions is presented in reference (17) and will be discussed further in this report in Section 5.

6. It should also be pointed out that the method presented here for the calculation of reliabilities assumes that the failure governing stress and strength distributions are constant or do not vary significantly from mission to mission. Consequently, the time dependency of the calculated reliabilities should be considered, since the Modified Goodman Diagram considers the strength aspect for one specific life duration only. This can be accomplished by generating a number of such diagrams, as is discussed later in this report.

Conclusions and Recommendations

The methodology presented in this chapter should make it possible to design components and products more scientifically and on a more sound engineering basis. The reliability approach of bridging the gap between the failure governing stress and strength provides a quantitative measure of the probability of success or failure of a component and of a product made up of such components.

It is recommended that every effort be expended in industry and in laboratories to gather failure governing stress and strength data and report them in a manner that would enable the development of their distributions, rather than averaging the measurements involved and discarding any which may appear too high or too low even though they are the result of well-controlled tests and experiments.

It is also recommended that design engineers use these methodologies and also contribute to their refinement. Furthermore, it is recommended that the designer think in terms of reliabilities, i.e., probabilities of success and unreliabilities, i.e., probabilities of failure, rather than in terms of the inadequate safety factor or safety margin.

This chapter has discussed the general approach to be used in designing by reliability. In the next chapters we will discuss first the proper failure governing strength theory to use for combined-stress fatigue, and second, show how this theory can be applied in designing parts subjected to combined-stress fatigue.

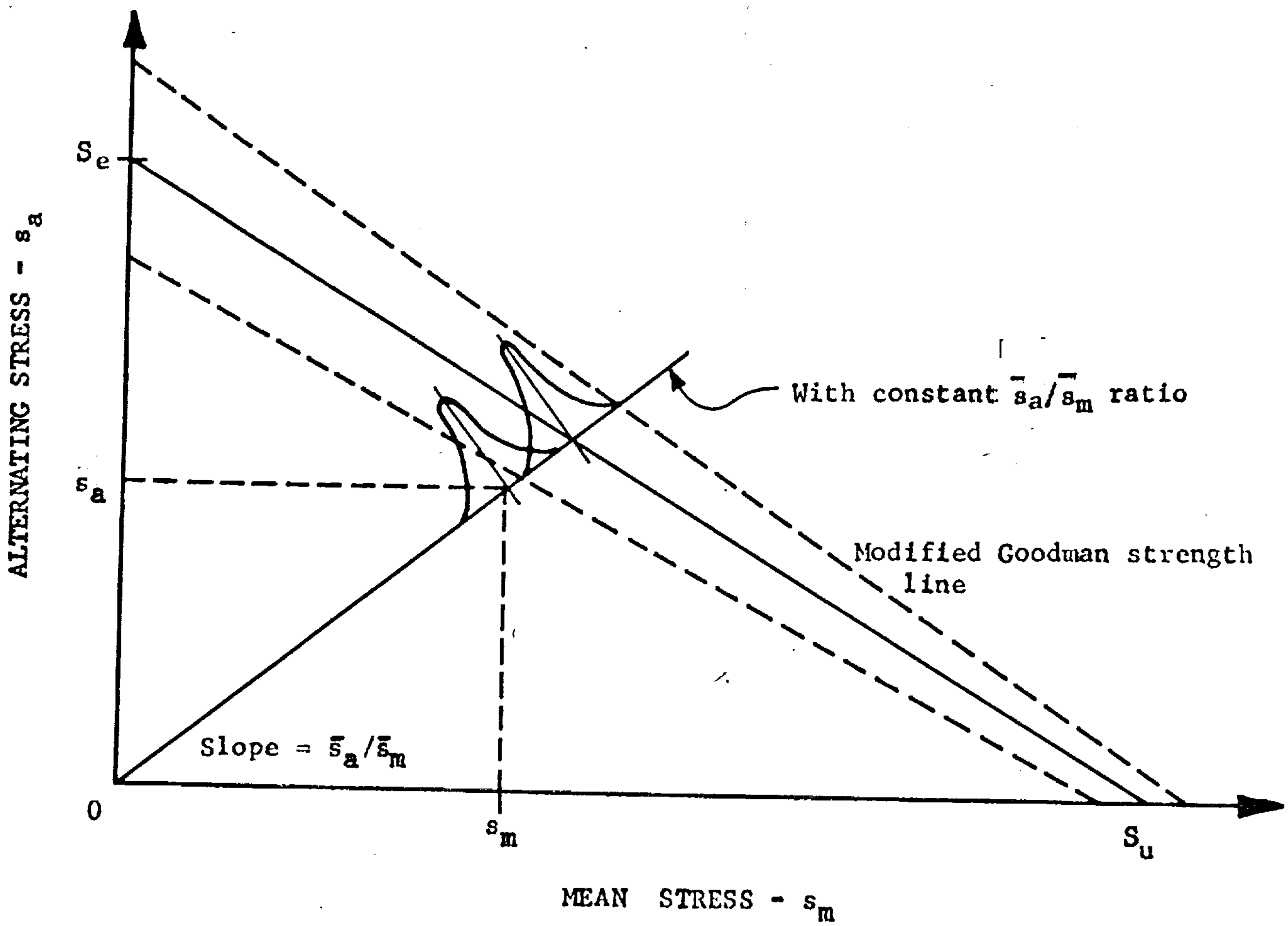


FIGURE 1.10. TRUE FAILURE GOVERNING STRESS AND STRENGTH DISTRIBUTIONS SUPERPOSED ON THE MODIFIED GOODMAN DIAGRAM.

CHAPTER 1.3

FAILURE GOVERNING STRENGTH THEORIES AND DESIGN METHODS FOR COMBINED-STRESS FATIGUE

Introduction

In this chapter the most recently available works dealing with combined-stress fatigue are presented to determine the best current methodology to use for design in these cases. One must realize from the outset that the conclusion and recommendations of this chapter are restricted to "alloy steels", and that different recommendations may apply to aluminum and cast iron.

A search of references (18) and (19) reveals no very recent work on the subject save that of Findley (20). Other fairly recent works (1953-1962) are summarized in (7), (14), (21), (22) and (23).

It is felt that all pertinent literature up to 1960 has been covered in references (7), (14), (21), (22) and (23). In particular, we have relied upon Smith (31). In February, 1963, Peterson (34) is content to accept Smith's work as covering the case of steady stress.

SAE 4340 is treated particularly in references (32), (33), and (35).

The evidence supporting the various combined-stress fatigue failure theories are given and specific recommendation for the case of alloy steels are made. The current design methodology for combined-stress fatigue, as outlined by Shigley (7) is discussed and supporting evidence for its adoption with some revisions, is given.

Theories of Failure

Several theories of failure have been proposed to deal with the static case, and each one has its application, depending usually on the type of material being considered. We shall summarize them here, and then consider which of them, if any, can be used for a theory of failure in fatigue. For the discussion, let P_1, P_2, P_3, be the amplitudes of the principal stresses where $P_1 \geq P_2 \geq P_3$ and let S be the alternating fatigue strength of the material. Then we have (22) the following failure criteria:

1. Maximum Principal Stress Criterion,

whereby if $P_1 \geq S$, failure results (1.3.1)

2. Maximum Shear Criterion,

whereby if $\frac{P_1 - P_3}{2} \geq S$, failure results (1.3.2)

3. Von Mises-Hencky or Octahedral Shear or Shear Strain Energy Criterion

whereby if $(P_1 - P_2)^2 + (P_2 - P_3)^2 + (P_3 - P_1)^2 \geq 2S^2$ failure results.

$$(1.3.3)$$

4. Maximum Principal Strain Criterion

whereby if $P_1 - \mu(P_2 + P_3) \geq S$, failure results. (1.3.4)

Here μ = Poisson's ratio

Each one of these equations can also be written in terms of the coordinate stresses σ_i and τ_{ij}.

To experimentally check which of these theories should be used, one can calculate the ratio of fatigue strength in torsion to fatigue strength in bending. The results are summarized in Table 1.1 (22).

TABLE 1.1

PREDICTED RATIO OF FATIGUE STRENGTH IN BENDING TO FATIGUE STRENGTH IN TORSION FOR VARIOUS THEORIES OF FAILURE

Theory of Failure	$\frac{\text{Fatigue Strength in Torsion}}{\text{Fatigue Strength in Bending}}$
Maximum Principal Stress	1.0
Maximum Shear Stress	0.5
Shear Strain Energy	0.577
Maximum Principal Stress	$\frac{1}{1+\mu}$

$$\frac{1}{1+\mu} = 0.77 \text{ for } \mu = 0.3$$

Results of experiments may now be plotted, as shown in Figures 1.11 and 1.12. In Figure 1.11, Forrest (22), (citing (14), (24), (25), (26), (27), (28), and (29), for steels), shows that the Von Mises criterion applies very well to steels. In Figure 1.12, Dolan (23), citing (24), (30), and others, shows much the same results although the maximum shear stress theory shows fairly good agreement also in many cases.

A different representation is shown in Figure 1.13, where Sines and Waisman (21) show the data of Sawert for steels. The agreement with the Von Mises theory is good. The maximum shear stress theory is seen to be conservative.

Dolan (23) shows the work of reference (27) and others in yet another form in Figure 1.14. Again, the agreement with the Von Mises theory is good for steels, and the maximum shear stress theory is seen to be conservative.

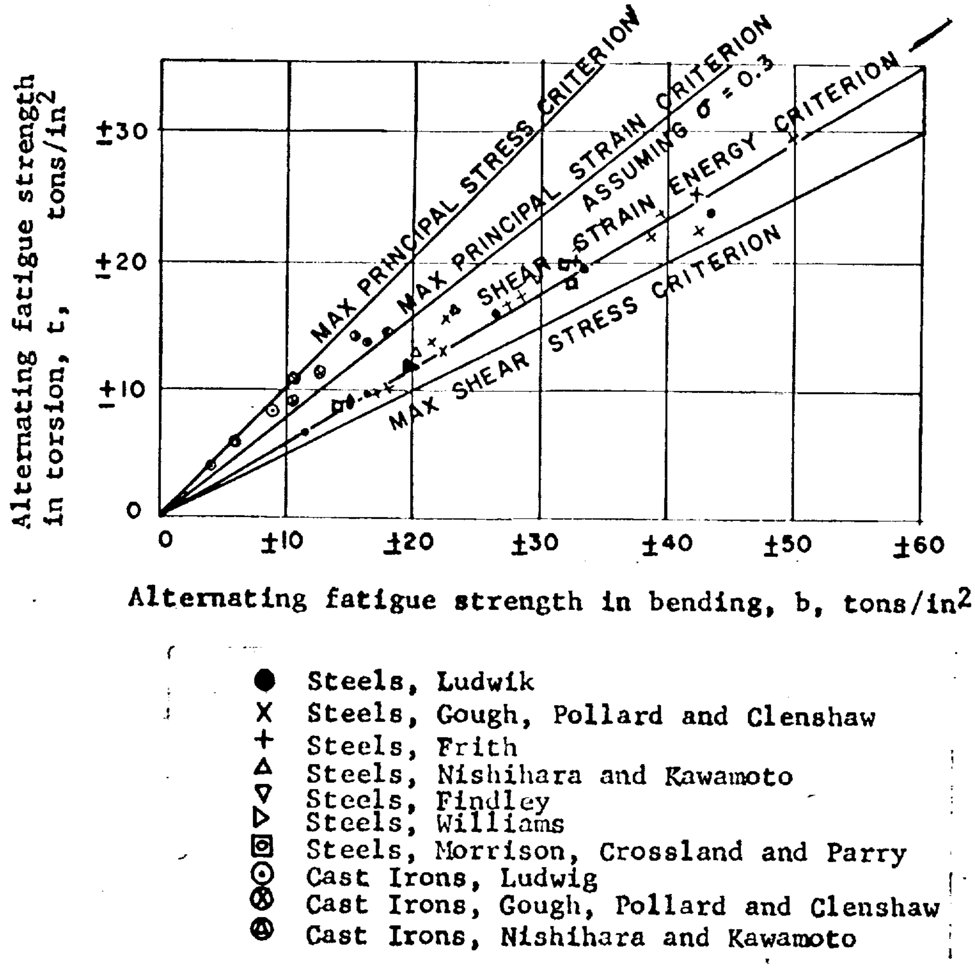


FIGURE 1.11 COMPARISON OF FATIGUE STRENGTHS IN BENDING AND TORSION FOR CAST IRON AND STEEL (REF. 22)

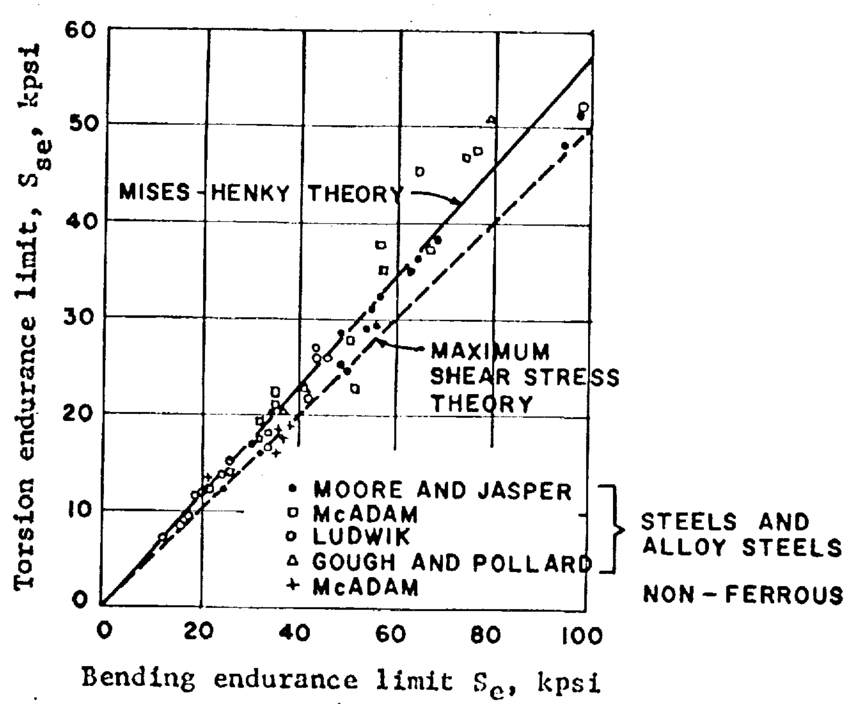
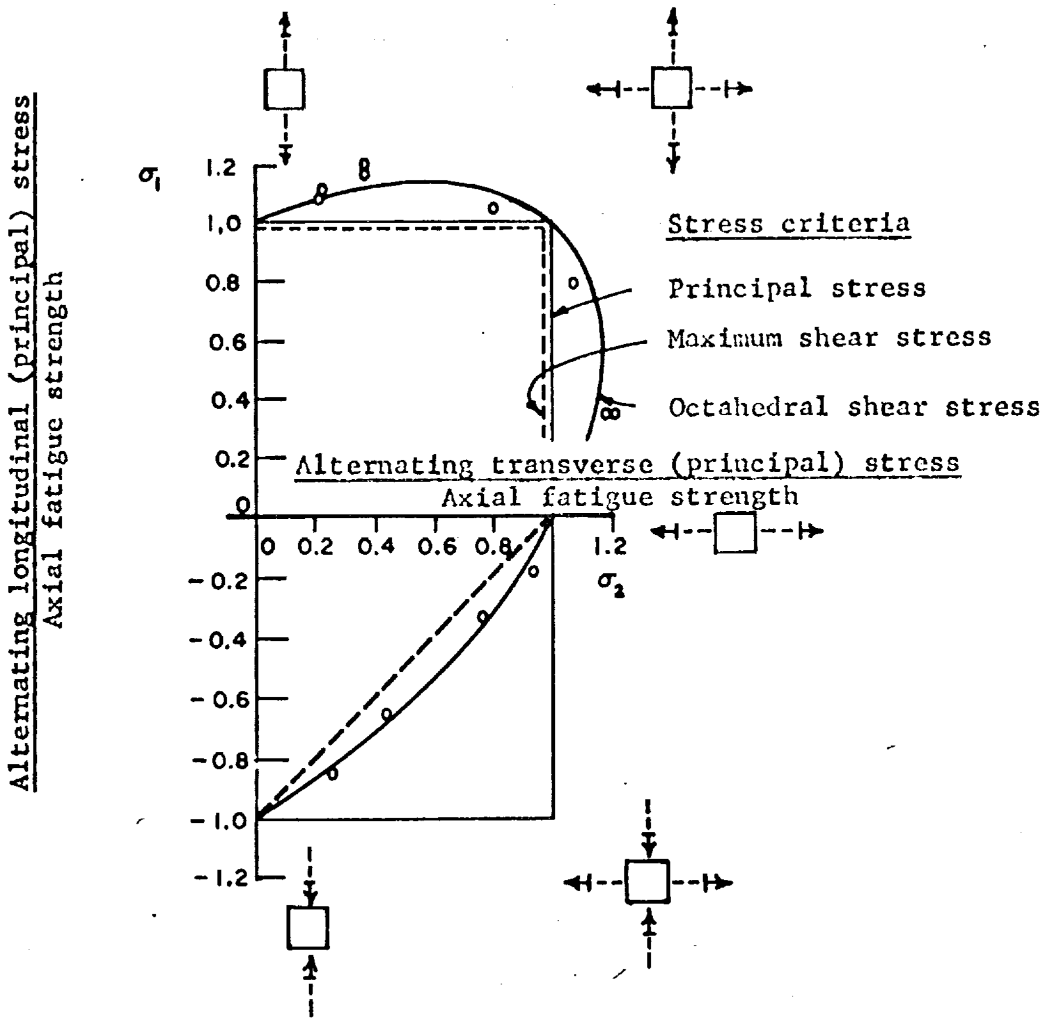


FIGURE 1.12 COMPARISON OF FATIGUE STRENGTH IN BENDING AND TORSION (REF. 23)



Annealed mild steel

FIGURE 1.13 FATIGUE STRENGTH AT 10^7 CYCLES OF REVERSED STRESS (REF. 21)

FIGURE 1.13

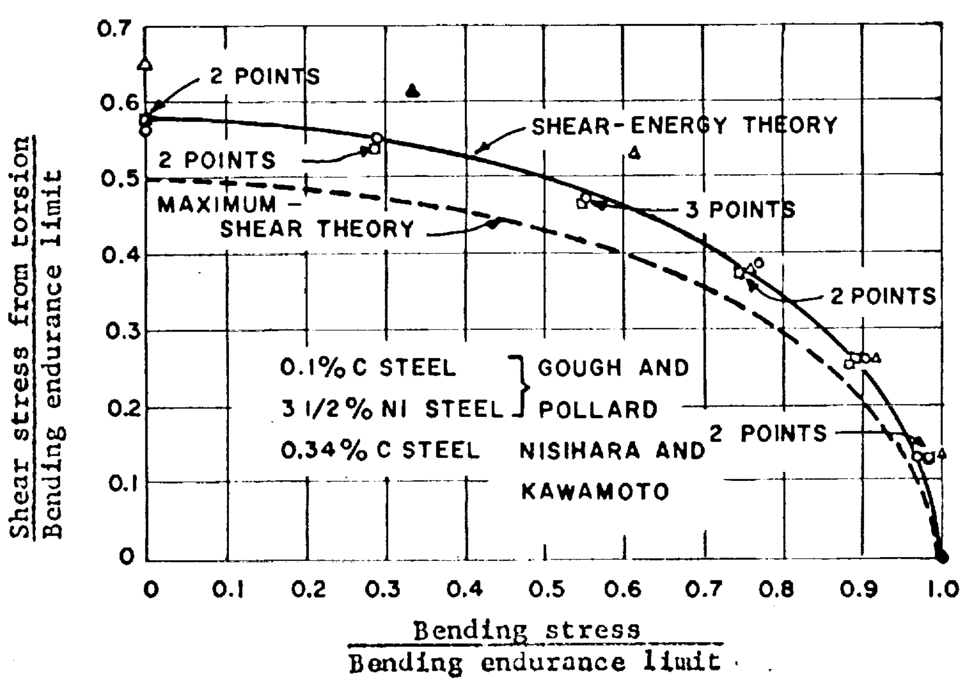


FIGURE 1.14 COMBINED-STRESS FATIGUE TEST DATA (REF. 23)

Another interesting representation is given by Sines and Waisman (21). Here, experimental results are compared against predicted values of ratio

$$\frac{\text{Max. shear stress in torsion test}}{\text{Max. shear stress in bending test}}$$

For the Von Mises theory, the predicted ratio is 1.15. As shown in Table 1.2, most alloy steels agree with this value, and in particular, the Ni-Cr-Mo steels (SAE 4340 alloy steels is 0.4%C, 1.8% Ni, 0.8% Cr, and 0.25% Mo.) are in close agreement.

TABLE 1.2 (ADAPTED FROM (4))

RATIO OF MAX. SHEAR STRESS IN TORSION TEST TO MAX. SHEAR STRESS IN BENDING TEST FOR VARIOUS MATERIALS

Material	$\frac{\text{Max. shear stress in torsion test}}{\text{Max. shear stress in bending test}}$
3 per cent Ni Steel	1.20
3-3 1/2 per cent Ni Steel	1.20
Cr-Va Steel	1.20
3 1/2 per cent Ni-Cr Steel, Normal Impact	1.305
3 1/2 per cent Ni-Cr Steel, Low Impact	1.27
Ni - Cr - Mo Steel, 60-70 ton	1.03 - 1.17
Ni - Cr - Mo Steel, 75-80 ton	1.04
Ni - Cr Steel, 95-105 ton	1.175

Conclusions and Recommendations

In view of the above evidence, it is recommended that the Von Mises failure governing strength criterion be adopted for steels with the maximum shear stress theory included as an alternate. It should be noted that none of these theories of failure can explain the phenomena of fatigue failure on a theoretical basis. In fact, all of the above theories have been developed to explain the phenomena of static, rather than dynamic failure. Therefore, if one wishes to adopt one of these theories for fatigue failures, which are dynamic in nature, the only justification for doing so is the fact that correlation does exist between the theoretical and the experimental results.

Design Method for Combined Stresses

Shigley (7) has presented a method for the design of such members based on the Von Mises theory of failure, which, in view of the conclusion above, is the primary choice. In Shigley's method, the stresses on the ordinary element shown in Figure 1.15 are used to solve for a mean stress and an alternating stress as

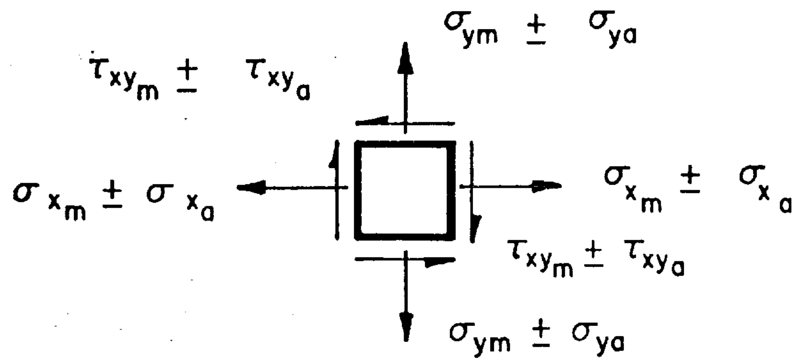


FIGURE 1.15. COMPONENT STRESSES FOR EQUATIONS (1.3.5) and (1.3.6)

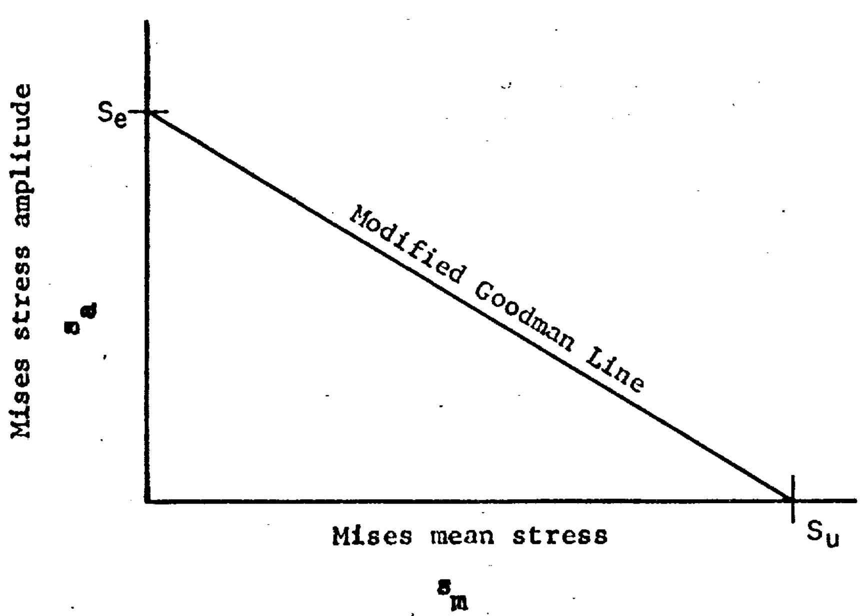


FIGURE 1.16. MODIFIED GOODMAN LINE

follows:

$$s_m = \sqrt{\sigma_{xm}^2 - \sigma_{xm}\sigma_{ym} + \sigma_{ym}^2 + 3\tau_m^2} \tag{1.3.5}$$

$$s_a = \sqrt{\sigma_{xa}^2 - \sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 + 3\tau_a^2} \tag{1.3.6}$$

These stresses are now used in connection with the Modified Goodman line of Figure 1.16.

The question now arises as to the applicability of the modified Goodman line for all cases of combined-stress fatigue. In particular, Sines and Waisman (21), citing Smith (31), show that the effect of a static torque on the alternating torsional fatigue strength is negligible until the torsional yield strength is exceeded. This is shown in Figure 1.17. Also, the effect of a static torque on the reversed bending fatigue strength shows the same trend (21) in Figure 1.18. Forrest (22, p. 104) gives the same result.

Nevertheless, the modified Goodman diagram of Figure 1.16 indicates that a reduction in fatigue strength should occur when a mean stress is superimposed on reversed bending.

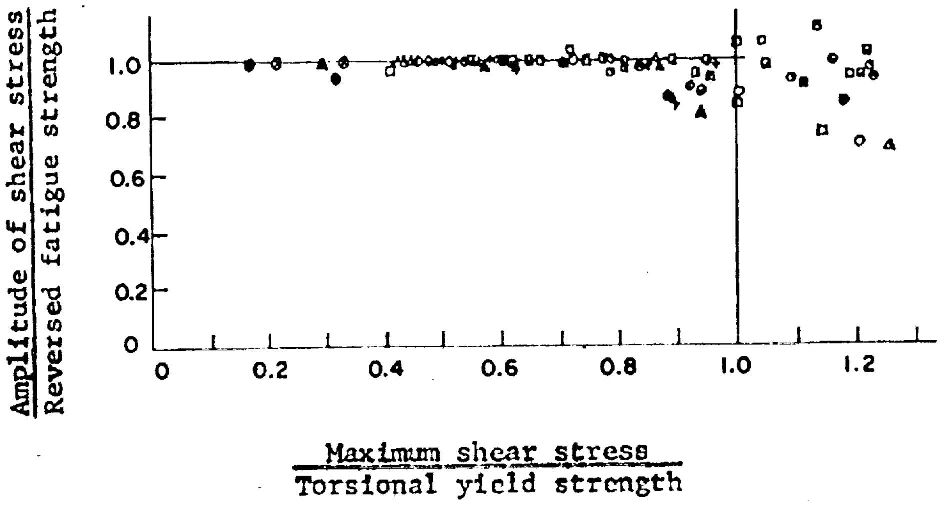
In the case of a rotating specimen when the mean stress is a shear stress induced by a shear stress induced by a steady torque and the alternating stress is a tensile-and-compressed stress induced by a bending moment, equations 1.3.5 and 1.3.6 become $s_m = 3\tau_m$ and $s_a = \sigma_{xa}$, whereby $\sqrt{3}\ S_s$ should be plotted along the abscissa and S_e along the ordinate of the modified Goodman diagram.

Further light may be shed on this problem when one realizes that the previous figures are for test results from smooth specimens. Forrest (22) gives in Figure 1.19 the results of Smith (31) which shows that for notched specimens the failure points fall along the modified Goodman line, if it is drawn from the endurance strength in shear to the torsional modulus of rupture. Lipson and Juvinall (8) have noted the same result, and state, "Since virtually all actual parts subjected to torsional loading contain stress raisers of some kind, the above phenomena is of little practical importance (referring to phenomena of Figures 1.17 and 1.18). For this reason the Goodman diagrams for torsional loading (drawn in (8), Chapter 22, p. 315) are of the conventional form, which agrees well with tests of notched torsional members."

Reference (32) gives results of actual tests on SAE 4340 steel, and it can be seen from Figure 1.20 that the test results for room temperature, notched and unnotched follow the modified Goodman line closely.

Recommendations for use of the modified Goodman diagram in specific cases of combinations of mean and alternating stresses are given in Table 1.3 and 1.4 based on the possible stress components shown in Figure 1.21.

When only alternating stresses are present, the equations to be used are given in Table 1.3. When mean stresses are present, the Goodman diagrams which should be drawn for each case are given in the Table 1.4. Note that Goodman diagrams are not always drawn from the point S_e to the point S_u. In some cases we find S_{se} on the ordinate and S_s on the abscissa. The proper procedure can



Material	Source
Cr-Ni steel, quenched tempered	
Siemens-Martin steel	P. Ludwik and J. Krystok
Baustahl, soft	
Schmieder bronze A, Rolled	
Aluminum alloy 17ST	Aluminum Company of America
Aluminum alloy 27ST	
Aluminum alloy 53ST	
Mild steel, hot rolled	
0.6%C steel, quenched tempered	
Si-Mn steel, quenched tempered	G. A. Hawkins
Cr-Va steel, quenched tempered	
Brass, 60 Cu-40Zn as rolled	
Copper, commercial pure, cold rolled	H. F. Moore and R. E. Lewis
Duralumin, as rolled	
Malleable iron, untempered	A. Pomp and M. Hempel
Malleable iron, tempered	
Beryllium bronze, 97.6%Cu - 2.4%Be	
0.9TC steel, quenched tempered	J. B. Johnson
Cr-Va steel, quenched tempered	
SAE3140 steel, quenched tempered	
SAE3140 steel, as hot rolled	
Tobin bronze, cold rolled	J. O. Smith
1.2%C steel, normalized	
3.5%Ni steel, special treatment "A"	H. F. Moore and T. M. Jasper
3.5%Ni steel, special treatment "D"	
0.49%C steel, normalized	
0.46%C steel, quenched drawn	D. J. McAdam Jr.

FIGURE 1.17

EFFECT OF MEAN TORSIONAL STRESS ON ENDURANCE LIMIT (Ref. 21)

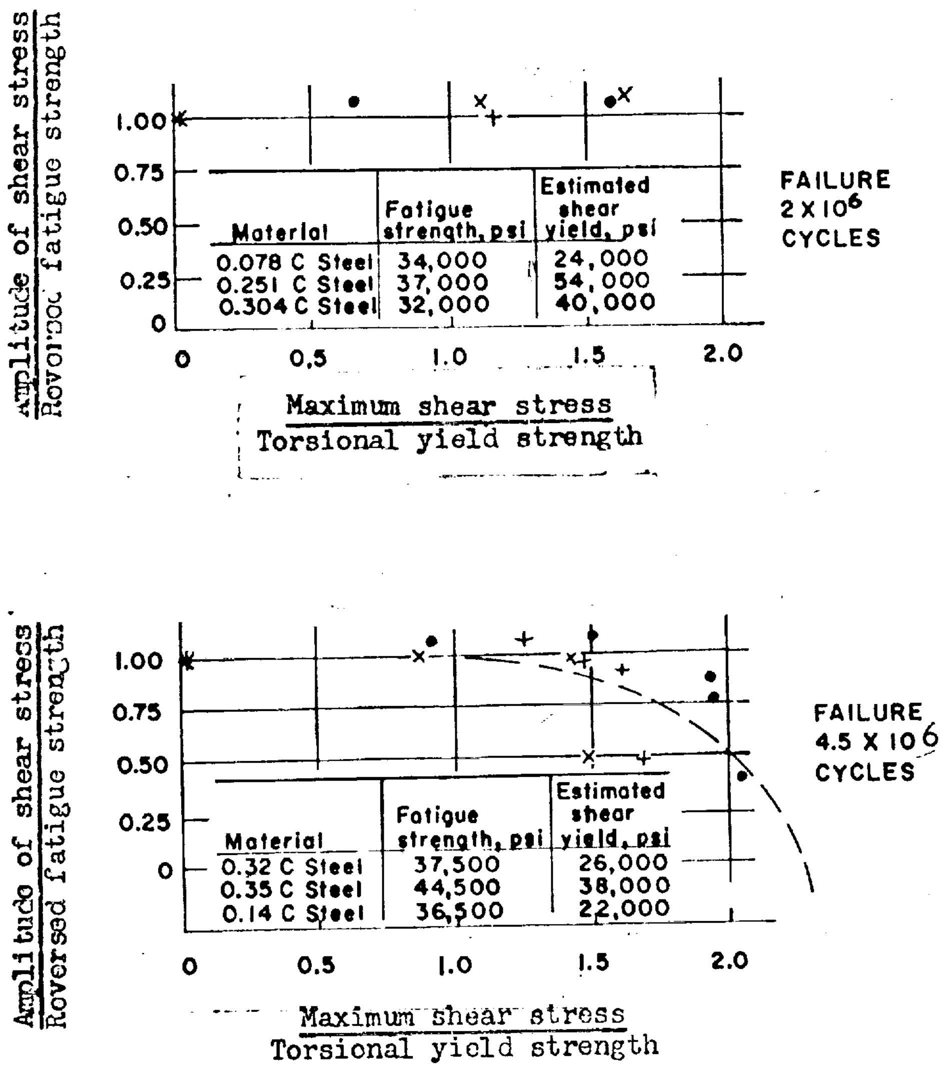
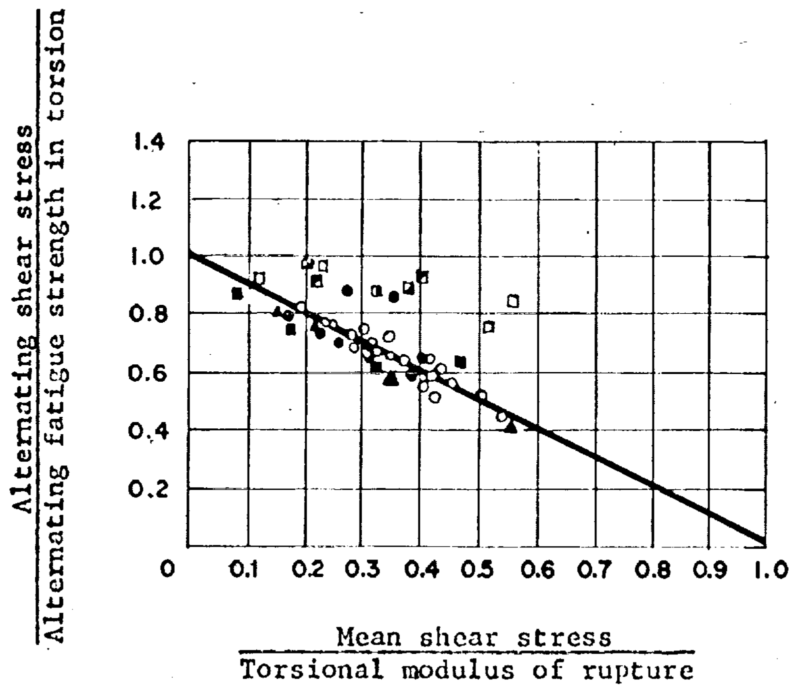


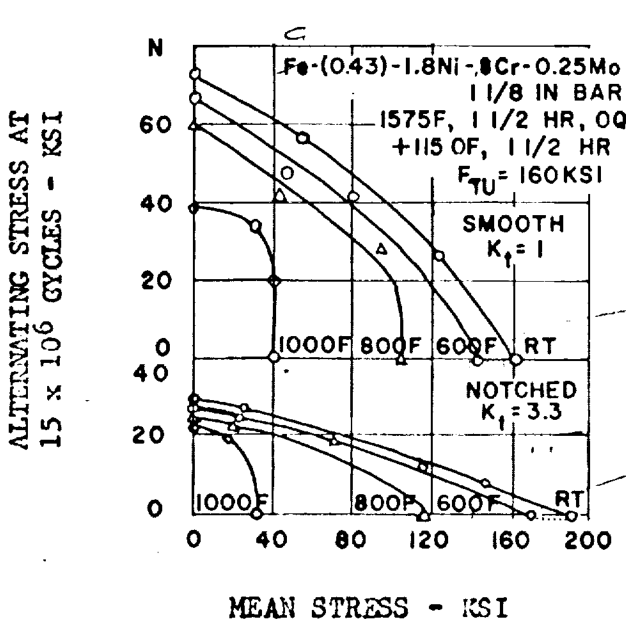
FIGURE 1.18

EFFECT OF MEAN TORSIONAL STRESS ON ENDURANCE LIMIT (Ref. 21)



■ SAE 3140 Heat treated steel, Notched (transverse hole) — J. O. Smith
▲ Tobin Bronze, Notched (transverse hole) — J. O. Smith
● 2 Spring steels, tested as coiled springs--Lea and Heywood
○ 9 Spring steels, tested as coiled springs--Zimmerli
□ Cr-Ni steel, Circumferential Notch — P. Ludwik and J. Krystok
◨ Cr-Ni steel, Corrosion — P. Ludwik and J. Krystok
◪ Malleable cast iron, untempered, surface as cast — A. Pomp and M. Hempel
◩ Malleable cast iron, tempered, surface as cast — A. Pomp and M. Hempel

FIGURE 1.19 NON-DIMENSIONAL ALTERNATING SHEAR STRESS-MEAN SHEAR STRESS DIAGRAM FOR NOTCHED SPECIMENS OF DUCTILE METALS IN TORSION (REF. 31)



$\frac{145}{160} = .907$

$\frac{175}{190} = .921$

FIGURE 1.20 STRESS RANGE DIAGRAMS FOR SMOOTH AND NOTCHED BAR AT ROOM TEMPERATURE TO 1000F (REF. 32)

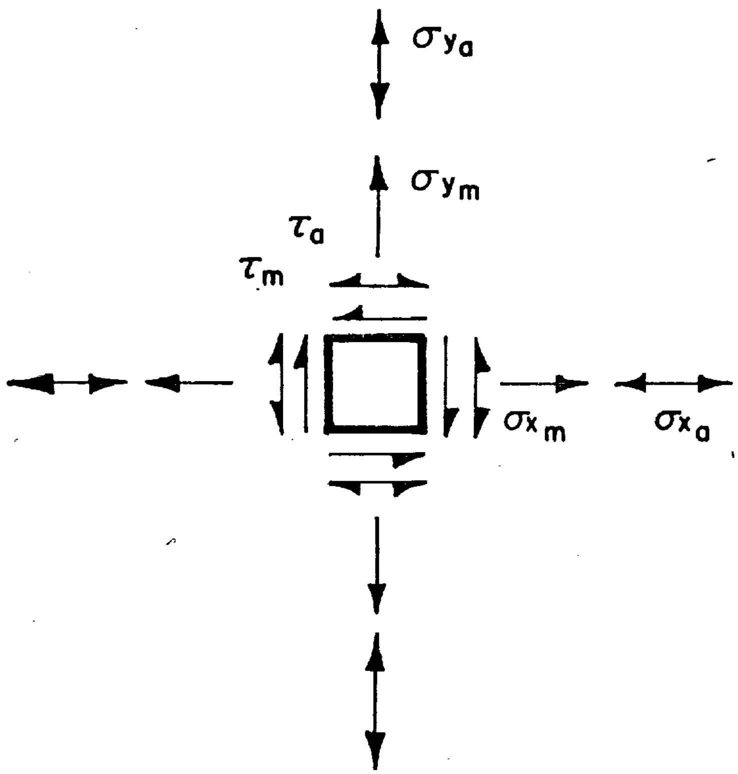


FIGURE 1.21. POSSIBLE STRESS COMPONENTS

be generalized in the following way:

> If in the equation for alternating stresses, only a τ_a term appears, the Goodman diagram should have $\sqrt{3}S_{se}$ on the ordinate. In other cases, plot S_e on the ordinate.
>
> If in the equation for mean stresses, only a τ_m term appears, plot $\sqrt{3}S_{s_u}$ on abscissa of the Goodman diagram. In other cases, plot S_{s_u} on the abscissa. It must be pointed out that $\sqrt{3}S_{se} \cong S_e$ and $\sqrt{3}S_{s_u} = S_u$ when the Von Mises theory of failure applies.

Summary

The Von Mises theory of failure is recommended for fatigue, with the maximum shear stress theory of failure as an alternate. The modified Goodman line is recommended as a conventional design methodology for combined stresses in fatigue, and it is felt that this modified Goodman line can also be adapted to design by reliability, as will be explained later.

As a result of this study, it is apparent that there are some differences from the standard Goodman diagram as it is usually drawn (7, ch. 5). The various stresses which may be present on a stress element are shown in Figure 1.21. The recommended procedure to be followed for each case is shown in Tables 1.3 and 1.4.

Now, with appropriate theories of failure and methods of attack firmly established, it is possible to proceed to Chapter 4 for the applications of the design reliability methodology to cases of combined stress fatigue.

TABLE 1.3 DESIGN EQUATIONS FOR SPECIFIC EXAMPLES OF GENERAL CASE SHOWN IN FIGURE 1.21 WHEN ONLY ALTERNATING STRESSES ARE PRESENT

STRESSES PRESENT	APPLICABLE EQUATIONS	GOODMAN DIAGRAM	JUSTIFYING REFERENCE(S)
CASE 1 σ_{xa} OR σ_{ya}	$\sigma_{xa} < S_e$ OR $\sigma_{ya} < S_e$	—	SHIGLEY
CASE 2 τ_a	$\tau_a < S_{se}$	—	SHIGLEY
CASE 3 σ_{xa} AND σ_{ya}	$\sigma_{xa}^2 - \sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 \leqslant S_e^2$	—	SHIGLEY
CASE 4 σ_{xa} AND σ_{ya} AND τ_a	$\sigma_{xa}^2 - \sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 + 3\tau_a^2 \leqslant S_e^2$	—	SHIGLEY
CASE 5 $\left\{\begin{matrix}\sigma_{xa} \text{ OR} \\ \sigma_{ya}\end{matrix}\right\}$ AND τ_a	$\sigma_{xa}^2 + 3\tau_a^2 \leqslant S_e^2$ OR $\sigma_{ya}^2 + 3\tau_a^2 \leqslant S_e^2$	—	SHIGLEY

NOTE: IF WE TAKE $S_{se} = 0.577 S_e$, THEN ALL OF THE ABOVE ARE SPECIAL CASES OF CASE 4.

TABLE 1.4 DESIGN EQUATIONS FOR SPECIFIC EXAMPLES OF GENERAL CASE SHOWN IN FIGURE 1.21 WHEN ALTERNATING AND MEAN STRESSES ARE PRESENT

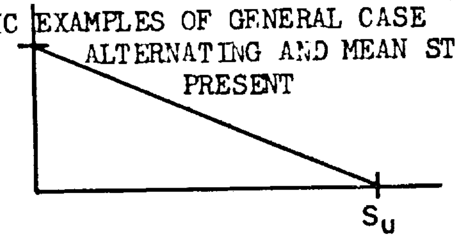
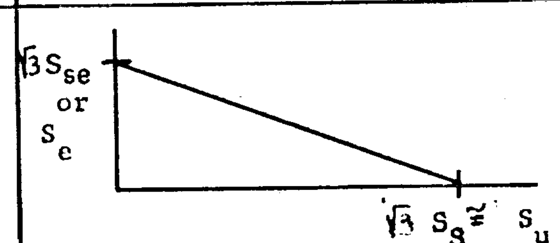
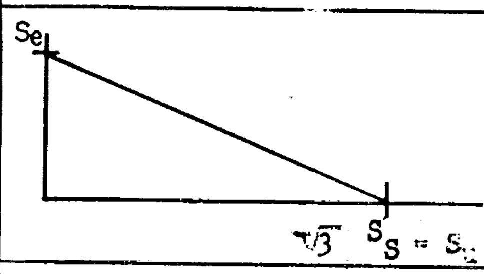
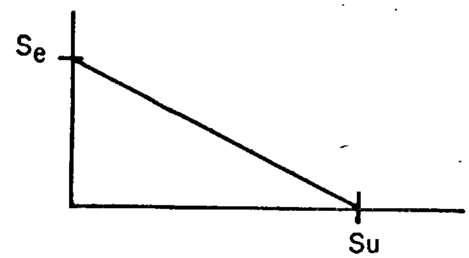
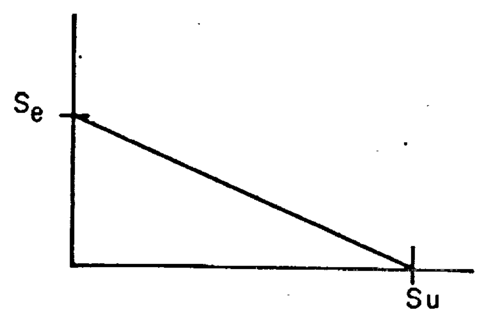
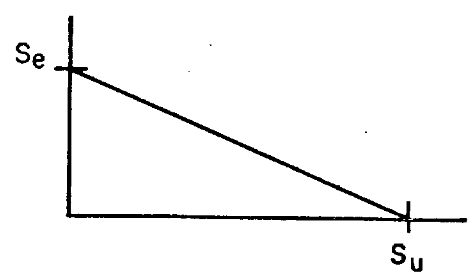
STRESSES PRESENT	APPLICABLE EQUATIONS	GOODMAN DIAGRAM	JUSTIFYING REFERENCE(S)
CASE 5 σ_{xa} & σ_{xm} OR σ_{ya} & σ_{ym}	$s_a = \sigma_{xa}$; $s_m = \sigma_{xm}$ OR $s_a = \sigma_{ya}$; $s_m = \sigma_{ym}$		SHIGLEY
CASE 6 τ_a & τ_m	$s_a = \sqrt{3}\tau_a$ $s_m = \sqrt{3}\tau_m$		FOREST, LIPSON & JUVINALL

TABLE 1.4
(CONT.)

STRESSES PRESENT	APPLICABLE EQUATIONS	GOODMAN DIAGRAM	JUSTIFYING REFERENCE(S)
CASE 7 σ_{xa} & τ_m OR σ_{ya} & τ_m	$s_a = \sigma_{xa}$; $s_m = \sqrt{3}\,\tau_m$ OR $s_a = \sigma_{ya}$; $s_m = \sqrt{3}\,\tau_m$		NONE (INFERRED FROM ABOVE CASE)
CASE 8 $\sigma_{xa}, \sigma_{xm},$ & τ_m OR $\sigma_{ya}, \sigma_{ym},$ & τ_m	$s_a = \sigma_{xa}$; $s_m = \sqrt{\sigma_{xm}^2 + 3\tau_m^2}$ OR $s_a = \sigma_{xy}$ $s_m = \sqrt{\sigma_{ym}^2 + 3\tau_m^2}$		SHIGLEY
CASE 9 σ_{xa}, σ_{xm} & τ_a, τ_m OR σ_{ya}, σ_{ym} & τ_a, τ_m	$s_a = \sqrt{\sigma_{xa}^2 + 3\tau_a^2}$ $s_m = \sqrt{\sigma_{xm}^2 + 3\tau_m^2}$ OR $s_a = \sqrt{\sigma_{ya}^2 + 3\tau_a^2}$ $s_m = \sqrt{\sigma_{ym}^2 + 3\tau_m^2}$		SHIGLEY
CASE 10 ALL	$s_a = \sqrt{\sigma_{xa}^2 - \sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 + 3\tau_a^2}$ $s_m = \sqrt{\sigma_{xm}^2 - \sigma_{xm}\sigma_{ym} + \sigma_{ym}^2 + 3\tau_m^2}$		SHIGLEY

The mean stress cases can be generalized by considering the following equations:

$$s_a = \sqrt{\sigma_{xa}^2 - \sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 + 3\tau_a^2} \qquad (1)$$

$$s_m = \sqrt{\sigma_{xm}^2 - \sigma_{xm}\sigma_{ym} + \sigma_{ym}^2 + 3\tau_m^2} \qquad (2)$$

If in eq. 1 , τ_a is the only component, plot $\sqrt{3}S_{se}$ on the alternating stress axis; otherwise S_e. If in eq. 2 τ_m is the only component, plot $\sqrt{3}S_s$ on the mean stress axis; otherwise Su. If τ_a and τ_m are the only existing stresses, then plot only S_{se} on the alternating stress axis and S_s on the mean stress axis, instead of $\sqrt{3}$ times these quantities as instructed before, because the two approaches will lead to the same results.

CHAPTER 1.4

APPLICATION OF THE DESIGN-BY-RELIABILITY METHODOLOGY TO COMBINED-STRESS FATIGUE

Introduction

The key to designing by reliability is to realize that all important design parameters must be treated as distributions. Values of stress, such as σ_{xa}, σ_{ya}, etc., and values of strength, such as S_e', S_e, S_{se}, etc., must be taken as distributions rather than single values. Once these modifications to conventional design methodology are made the basis for the design by reliability methodology is established.

The distributions of these quantities may not be easy to find. Sections 3 and 4 of this report will be devoted to describing in detail how such distributions may be obtained. In this chapter, in order to develop the methodology further, we will assume that the distributions are given.

The fatigue problem in general may be broken into the time-invariant and the time-variant cases. In Fig. 1-9, the portion of the S-N curve to the right of the knee @ 10^6 cycles is assumed to be a region of time-invariant strength, that is $\bar{S}_a$ and σ_{sa} are assumed to be constant in this region. However, the region to the left of the knee of the curve is a time-variant region; that is, the mean and standard deviation of the strength distribution vary with time.

The illustration of the design by reliability methodology will begin with some examples of the time-invariant case.

The Time-Invariant Case

Example 1.1

The first case is that of a part subjected only to reversed bending. From Figure 1.21, the following stress components would result:

$$\sigma_{xa} \neq 0$$

$$\sigma_{xm} = \sigma_{ym} = \sigma_{ya} = \tau_a = \tau_m = 0 .$$

This is case 1 in Table 1.3. Assume that the endurance limit of this part is known from laboratory tests and is given by a normal distribution with

$$\bar{S}_e = 50{,}000 \text{ psi}$$

$$\sigma_{S_e} = 3{,}000 \text{ psi}$$

Further, take the case where the applied stress can be described by

$$\bar{s} = 30{,}000 \text{ psi}$$

$$\sigma_s = 5{,}000 \text{ psi}$$

Then from equations 1.2.24b, and 1.2.24c,

$$\bar{\xi} = 50{,}000 - 30{,}000 = 20{,}000 \text{ psi}$$

$$\sigma_{\xi} = \left[(3)^2 + (5)^2\right]^{1/2} \times 1{,}000$$

$$= (5.83)\ (1{,}000)$$

$$= 5{,}830 \text{ psi}$$

Now,

$$t = -\frac{\bar{\xi}}{\sigma_{\xi}} = -\frac{20{,}000}{5{,}830} = -3.43$$

For which

$$R = 0.9^3964$$

Figure 1.22 depicts the stress and strength distributions for this part.

Example 1.2

Take now the case of reversed bending combined with a steady torque.

Referring to Fig. 1.21, the following stress components would result:

$$\sigma_{xa} \neq 0$$

$$\tau_m \neq 0$$

$$\sigma_{xm} = \sigma_{ym} = \sigma_{ya} = \tau_a = 0.$$

This corresponds to Case 7 in Table 1.4, therefore we will use the Goodman diagram for Case 7, but we must now treat the stresses and strengths as distributions rather than as single values.

Figure 1.23 depicts the failure governing stress and failure governing strength distributions. The Goodman diagram is taken conservatively to be a straight line joining the distributions for S_e and $\sqrt{3}S_s = S_u$. On this conservative basis an estimate of the lower limit of the actual reliability is obtained. The expected value of the actual reliability will be determined when the true strength distribution is generated from the experimental results of this research.

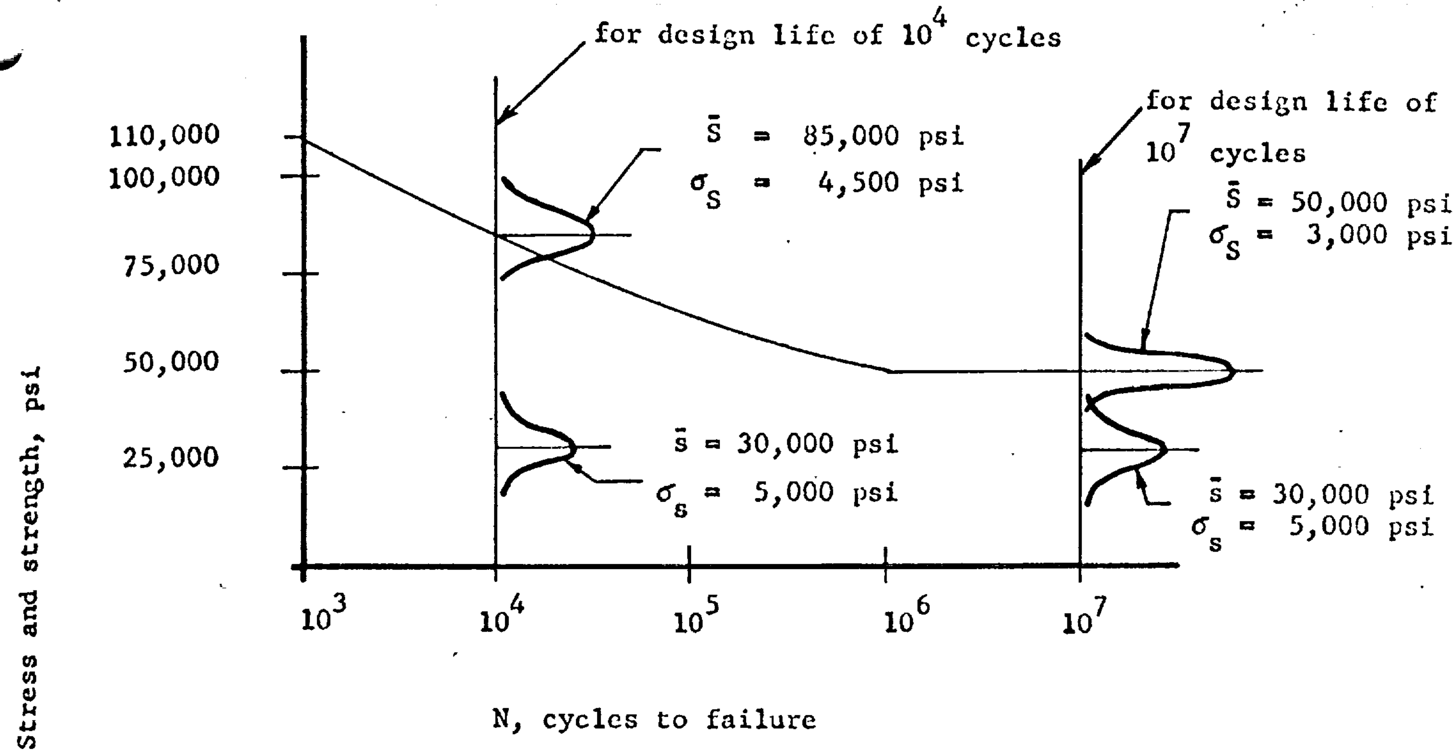


Figure 1.22 S-N DIAGRAM WITH STRESS AND STRENGTH DISTRIBUTIONS FOR EXAMPLES 1.1 AND 1.4

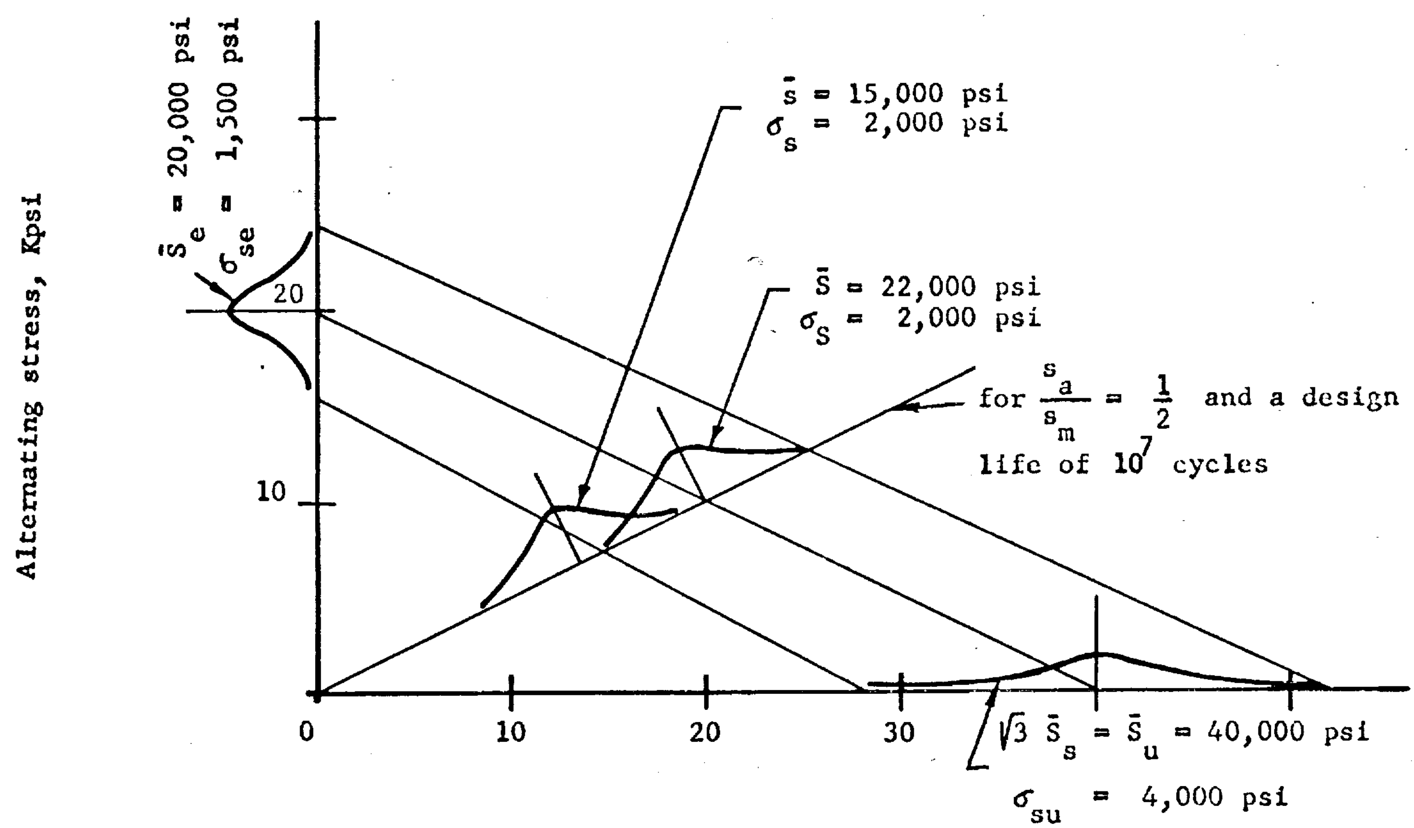


Figure 1.23 GOODMAN DIAGRAM WITH STRESS AND STRENGTH DISTRIBUTIONS FOR EXAMPLE 1.2

The distribution for the failure governing strength, S, can be estimated from this linear assumption by scaling off from Figure 1.23, and it is given by

$$\bar{S} = 22,000$$

$$\sigma_S = 2,000$$

The distribution of the failure governing stress in this case is not easily determined. Note that this distribution will be a combination of the distributions of $s_a = \sigma_{xa}$ and $s_m = \sqrt{3}\,\tau_m$. Let us assume for the present that the failure governing stress distribution is a normal distribution with

$$\bar{s} = 15,000 \text{ psi}$$

$$\sigma_s = 2,000 \text{ psi}$$

at a stress ratio of $\frac{s_a}{s_m} = 1/2$, as shown in Figure 1.23.

Then from equations 1.2.24b and 1.2.24c,

$$\bar{\xi} = 22,000 - 15,000 = 7,000 \text{ psi}$$

$$\begin{aligned}\sigma_\xi &= \left[(2)^2 + (2)^2\right]^{1/2} \times 1,000 \\ &= (2.83) \times 1,000 \\ &= 2,830 \text{ psi}\end{aligned}$$

Now,

$$t = -\frac{\bar{\xi}}{\sigma_\xi} = -\frac{7,000}{2,830} = -2.46$$

from which

$$R = 0.9861062$$

Note that the reliability is relatively low. In this case the factor of safety for the part is

$$\text{F.S.} = \frac{22,000}{15,000} = 1.47$$

which is also relatively low.

Example 1.3

Consider now the case of a part which is subjected to an alternating σ_x, a mean σ_x, and a mean and alternating τ. Then we have

$$\sigma_{xa} \neq 0$$

$$\sigma_{xm} \neq 0$$

$$\tau_a \neq 0$$

$$\tau_m \neq 0$$

$$\sigma_{ya} = \sigma_{ym} = 0$$

This is Case 9 of Table 1.4.

The failure governing strength is again taken conservatively to be a linear relation between the alternating stress failure governing strength and the mean stress failure governing strength. In this case, the appropriate alternating strength is S_e, the endurance limit of test specimens which are subjected to reversed bending alone. The appropriate mean, or static, strength is the static ultimate strength of the material.

The failure governing stress can be found by synthesizing the various components of stress the part is subjected to. If the Von Mises-Hencky (distortion energy) theory of failure is used, then the appropriate failure governing stress formulas are

$$s_{a_1} = \sqrt{\sigma_{xa}^2 + 3\tau_a^2} \qquad (1.4.1a)$$

$$s_{m_1} = \sqrt{\sigma_{xm}^2 + 3\tau_m^2} \qquad (1.4.1b)$$

If the maximum shear stress theory of failure is used, the appropriate equations are

$$s_{a_2} = \sqrt{\left(\frac{\sigma_{xa}}{2}\right)^2 + \tau_m^2} \qquad (1.4.2a)$$

$$s_{m_2} = \sqrt{\left(\frac{\sigma_{xm}}{2}\right)^2 + \tau_m^2} \qquad (1.4.2b)$$

Strictly speaking, design by reliability would call for the synthesis of the distributions of σ_{xa}, τ_a, σ_{xm}, and τ_m as functions of random variables by use of either equations 1.4.1 or 1.4.2. However, let us defer the discussion of this rather difficult problem until the next Section, and assume that we can, for the present, combine the means of the stress components, and assume that the resulting distribution is normal, with a standard deviation equal to 15 per cent of the mean.

Choosing some values for purposes of illustration, let

$$\sigma_{xa} = 16{,}000 \text{ psi}$$

$$\tau_a = 8{,}000 \text{ psi}$$

$$\sigma_{xm} = 10,000 \text{ psi}$$

$$\tau_m = 2,000 \text{ psi}$$

We will consider two solutions, one using the Von Mises-Hencky theory of failure and one using the maximum shear stress theory of failure.

Von Mises-Hencky Theory of Failure Solution

From equations 1.4.1

$$s_{a_1} = \sqrt{(16)^2 + 3(8)^2} \times 1,000$$

$$= 21,170 \text{ psi}$$

$$s_{m_1} = \sqrt{(10)^2 + 3(2)^2} \times 1,000$$

$$= 10,580 \text{ psi}$$

Plotting the above on Figure 1.24, the mean of the failure governing stress is given by

$$\bar{s}_1 = \sqrt{s_{a_1}^2 + s_{m_1}^2} = \sqrt{(21.17)^2 + (10.58)^2} \times 1,000 = 23,600 \text{ psi}$$

Taking its standard deviation as 15 percent of the mean,

$$\sigma_{s_1} = 3,540$$

The failure governing strength distribution is obtained by graphical, linear interpolation from Fig. 1.24 as

$$\bar{S} = 40,000 \text{ psi}$$

$$\sigma_S = 2,000 \text{ psi}$$

From equations 1.2.24b and 1.2.24c,

$$\bar{\xi} = 40,000 - 23,600 = 16,400 \text{ psi}$$

$$\sigma_\xi = \sqrt{(2)^2 + (3.54)^2} \times 1,000$$

$$= 3,800 \text{ psi}$$

and

$$t = -\frac{\bar{\xi}}{\sigma_\xi} = -\frac{16,400}{3,800} = -4.32$$

from which

$$R = 0.9^4 8439$$

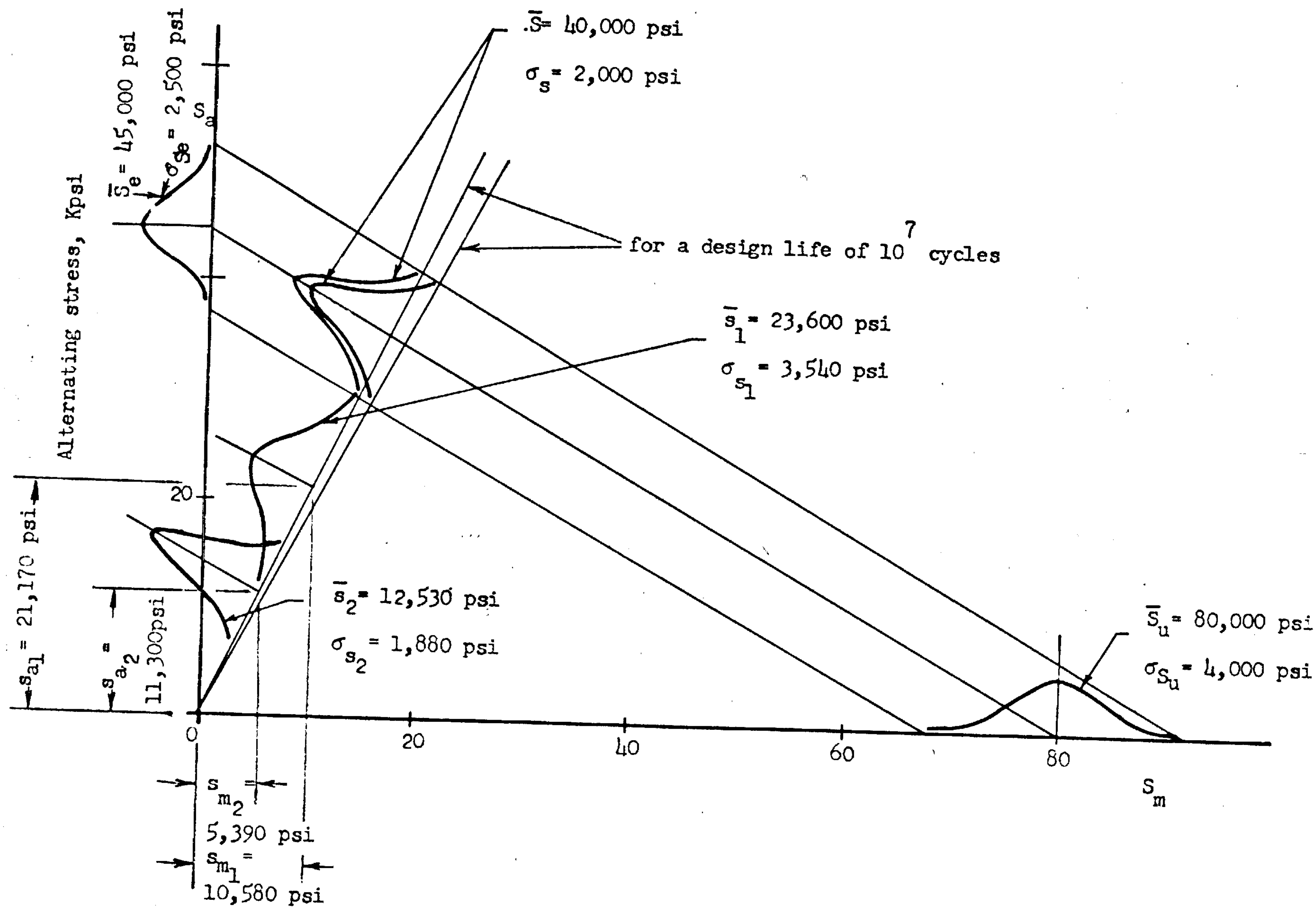


Figure 1.24 MODIFIED GOODMAN DIAGRAM FOR EXAMPLE 1.3

Maximum Shear Stress Theory of Failure Solution

The failure governing stress is now given by equations 1.4.2 as

$$s_{a_2} = \sqrt{\left(\frac{16}{2}\right)^2 + (8)^2} \times 1,000$$

$$= 11,300 \text{ psi}$$

$$s_{m_2} = \sqrt{\left(\frac{10}{2}\right)^2 + (2)^2} \times 1,000$$

$$= 5,390 \text{ psi}$$

The failure governing stress mean is then given by, as in the previous case,

$$\bar{s}_2 = \sqrt{(11.3)^2 + (5.39)^2} \times 1,000 = 12,530 \text{ psi}$$

The standard deviation is again taken to be 15 percent of the mean or

$$\sigma_{s_2} = 1,880 \text{ psi}$$

The failure governing strength distribution; as obtained from Figure 1.24, is given by

$$\bar{S} = 40,000 \text{ psi}$$

$$\sigma_S = 2,000 \text{ psi}$$

From equations 1.2.19b and 1.2.19c,

$$\bar{\xi} = 40,000 - 12,530 = 27,470$$

$$\sigma_{\xi} = \sqrt{(2)^2 + (1,88)^2} \times 1,000 = 2,750 \text{ psi}$$

and

$$t = \frac{\bar{\xi}}{\sigma_{\xi}} = -\frac{27,470}{2,750} = -10+$$

from which

$$R = 0.9^{15}+$$

Interestingly enough the maximum shear stress theory is not conservative for this example.

These examples should serve to illustrate the design-by-reliability methodology for the case of "infinite life" design. The other combination of stresses which may arise as special cases of the stresses illustrated in Figure 1.21 can be treated in a similar fashion.

Now we shall proceed to discuss some examples for the time-variant, or "finite life", case.

The Time-Variant Case

The main difference between the time-invariant and the time-variant case is that for the former, the appropriate endurance limit, S_e, of the material is used, while for the latter the finite-life fatigue strength, S, of the material is used. In Figure 1.22, this distribution is shown for 10^4 life cycles. The generation of such a finite-life distribution will be considered in Section 3 of this report.

Example 1.4.

Consider Example 1.1, but instead of designing for infinite life, let us design for a finite life of 10^4 cycles. As shown in Fig. 1.22, let us take

$$\bar{S} = 85{,}000 \text{ psi}$$

$$\sigma_S = 4{,}500 \text{ psi}$$

$$\bar{s} = 30{,}000 \text{ psi}$$

$$\sigma_s = 5{,}000 \text{ psi}$$

Then,

$$t = \frac{\bar{\xi}}{\sigma_\xi} = -\frac{85{,}000 - 30{,}000}{\sqrt{(4.5)^2 + (5)^2} \times 1{,}000} = -\frac{55{,}000}{6{,}730} = -8.17$$

and

$$R = 0.9^{15}$$

This compares with $R = 0.9^3 3964$ in Example 1.1 for a life of 10^7 cycles.

If such a high reliability is not desired and the target reliability for this part is $R = 0.9995$, then the mean applied stress can be increased to $\bar{s} = 108{,}200$ psi. This may be calculated as follows:

If $R = 0.9995$ then $t = -\frac{\bar{\xi}}{\sigma_\xi} = -3.481$, from normal distribution area tables. Assuming the strength distribution remains the same, as it should, and the stress variability remains the same, then

$$-3.481 = \frac{85{,}000 - \bar{s}}{6{,}730}$$

or

$$\bar{s} = 108{,}200 \text{ psi}$$

This says that the same part may be designed to much higher stresses, hence with smaller sections, lighter weight, and presumably lower cost. Hence the great value of designing by reliability.

The design by reliability procedure for finite life when mean stresses are involved is analogous to that of infinite life. However, instead of using the distribution of S_e on the ordinate of the Goodman diagram as we did in Example 1.2 and Fig. 1.23, we use the appropriate finite-life fatigue strength distribution.

Example 1.5

Consider again the part of Example 1.2, but let us design it for a finite life of, say 10^4 cycles. Now the Goodman diagram will have the finite-life fatigue strength determined by tests imposing reversed bending and steady torque on the specimen, plotted on the ordinate.

Referring to Fig. 1.26, the strength distribution is

$$\bar{S} = 26,800 \text{ psi}$$

$$\sigma_S = 2,300 \text{ psi}$$

The stress distribution is taken to be the same as that of Example 1.2, or

$$\bar{s} = 15,000 \text{ psi}$$

$$\sigma_s = 2,000 \text{ psi}$$

Then,

$$t = -\frac{\bar{\xi}}{\sigma_\xi} = -\frac{26,800 - 15,000}{\sqrt{(2.3)^2 + (2)^2} \times 1,000} = -\frac{11,800}{3,100} = -3.82$$

and,

$$R = 0.9^3 8665$$

This compares with a R = 0.9861062 in Example 1.2 and shows an increase in reliability from the infinite life case, as would be expected, because we wish the part to last a substantially fewer number of cycles.

It should be obvious by now that to treat Example 1.3 as a finite life, or time variant problem, one would need to use the finite life endurance strength at the desired number of cycles on the ordinate of Fig. 1.24 rather than the endurance limit. The the solution would proceed the same as in Example 1.3.

Summary

In this section, the differences, and also the similarities, in the design by reliability methodology as compared to the conventional design methodology, have been pointed out. Conventional design methodology forms a framework for the design by reliability methodology once it is realized that in designing by reliability the important parameters of the design should be treated as distributions rather than as unique values.

The various theories of failure for metals have been reviewed and the distortion energy theory of failure has been shown to apply best to alloy steels, with the maximum shear stress theory of failure as an alternate.

Several illustrative examples which highlight the concept of designing by reliability have been given. However, several important questions still must be answered, such as:

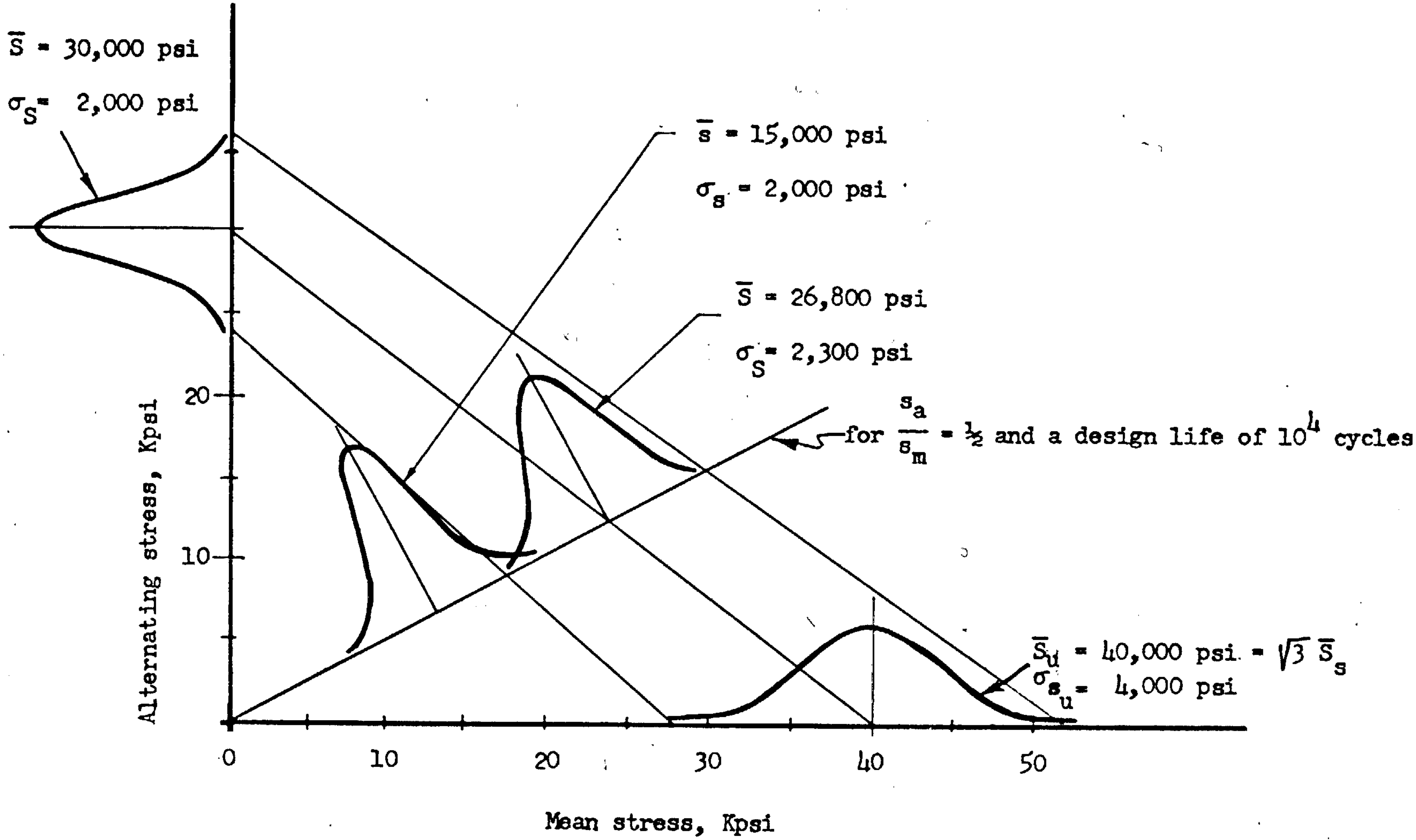


Figure 1.26 GOODMAN DIAGRAM OF STRESS AND STRENGTH DISTRIBUTIONS FOR EXAMPLE 1.5

1. How can the failure governing stress and strength distributions be obtained in actual design practice?

2. How can conventional engineering formulas be used when distributions, rather than unique values, are involved?

3. How is reliability found when the failure governing stress and the failure governing strength distributions are not normal?

These, and other questions, are discussed in the following Sections. In some cases, answers are given; in others, the answers must await further research and development on the methodology.

One vital area which will be discussed next in Section 2, is that of determining functions of distributions.

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SECTION 2

FUNCTIONS OF RANDOM VARIABLES FOR STRUCTURAL RELIABILITY APPLICATIONS

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CHAPTER 2.1

INTRODUCTION

This section presents the results of a literature search and theoretical study which has been conducted so that the widely scattered results of the work on functions of random variables which are thought to be important for structural reliability applications can be assembled and organized.

Freudenthal (1, p. 322)* has pointed out that it is futile to try to decide on the basis of data analysis which of the many distributions best describe a structural design quantity, such as fatigue life. He recommends that such distributions be selected for study on the basis of physical reasoning. Therefore, one portion of this section was devoted to studying works such as (2-6) which do attempt to define or select significant distributions for structural reliability applications. The distributions given in Chapter 2.2 were selected, and they are discussed further in that chapter.

Chapter 2.3 contains the mathematical definitions of the distributions selected, with comments about their properties and parameters.

In Chapter 2.4, mathematical concepts and definitions are presented. These are basic statistical mathematics needed for understanding the work in this section. Then the mathematical methods which are important for the analysis of functions of random variables are presented in Chapter 2.5. These include:

1. Algebra of Normal Functions Method
2. Change of Variable Method
3. Moment Generating Function Method
4. Fourier Transform, Convolution, and Inversion Method
5. Mellin Transform, Convolution, and Inversion Method
6. Characteristic Function Method
7. Cumulative Distribution Function Method
8. Monte Carlo Method

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*Numbers in parenthesis refer to REFERENCES at the end of this section.

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Chapter 2.6 presents some mathematical derivations and significant results for the more important functions, and utilizes the techniques which are thought to have the most promise for future studies. In particular, the method involving the Mellin transform (7), (8) is believed to have great potential for treating products and quotients of random variables because of its suitability to both analytical and digital computer solution (7).

Listed here are some of the results which had not come to the authors' attention before the beginning of this current study and which are thought to be important to the area of structural reliability:

1. The p d f of products of n independent normal variables $N(0,\sigma)$* for $n < 10$, (7).

2. The p d f of the quotient of two independent gamma variables (7).

3. The product of two independent beta variables of the first kind (9).

4. An extensive work on the quotients of normal variables (10).

5. The result for $1/x$ where x is lognormal, or $L(\mu,\sigma)$, (3).

6. The product of two lognormal variables (3).

7. The quotient of two lognormal variables (3).

8. The result for cx^b where x is $L(\mu,\sigma)$, (3).

This summarizes some of the highlights.

In the literature of statistical theory, one finds much work on functions of random variables which is quite useful in statistical problems, but is not especially suited to solving problems in structural reliability (11-13). Therefore this section represents an attempt to deal with functions of random variables which will be useful in solving structural reliability problems.

As an example, consider the following situation. The endurance strength of a part in actual service, subject to fatigue loading, is given by (14, p. 166)

$$S_e = S_e' \, k_a \, k_b \, \ldots . . k_n \qquad (2.1.1)$$

---

*The symbol $N(\mu,\sigma)$ stands for Normal distribution with mean μ and standard deviation σ. See the List of Symbols for other symbols.

where:

S_e = endurance strength of the part in service.

S_e' = endurance strength of the material from which the part is made, based on laboratory tests on a standard specimen.

$k_a,\ldots,k_n$ = factors which account for the differences between laboratory conditions and service conditions.

Structural reliability theory demands that all of these quantities be treated as distributions. The question now becomes, given the distribution of S_e', and of the $k_a,\ldots,k_n$, what is the distribution of the strength of the part, S_e? Once this strength distribution is known, it can be combined with the stress distribution to find reliability as described in Ref. (15). This poses the problem of finding the distribution of the product of a number of distributions which are not necessarily identically distributed.

Consider another problem. The safety factor can be defined on a statistical basis in the following way (16, p. 9).

$$\nu = \frac{S}{s} \tag{2.1.2}$$

where:

ν = the safety factor
S = the strength of the part or structure
s = the stress on the part or structure

and the three variables are all distributed. Then the probability of failure is defined as

$$\text{P. F.} = P_r(\nu < 1) \tag{2.1.3}$$

Thus, once the problem of the quotient of the two random variables S and s has been solved, the reliability of the part or structure can be evaluated.

It is easy to see that all structural formulas, such as

$$s = \frac{P}{A} \tag{2.1.4}$$

$$s = \frac{Mc}{I} \tag{2.1.5}$$

and so on ad infinitum can be put on the basis of all of the governing loads, dimensions, and factors being considered as distributions rather than as single values.

Another problem which might be treated by application of the functions of random variables is that of MRB (Materials Review Board) action. Frequently an engineer in practice is called upon to decide what is to be done with parts which are slightly out of blueprint tolerance. Take the simple example of a round rod in axial tension. From equation (2.1.4),

$$s = \frac{P}{A} = \frac{P}{(\pi/4)d^2} \tag{2.1.6}$$

Suppose now that some of the pieces are found to be of smaller diameter than specified by the blueprint. The engineering representative to MRB must recommend as to the disposition of these parts. Currently, this is not done on a rational basis; however, by use of functions of random variables, the engineer can quantitatively evaluate the effect of a certain number of undersized parts on the distribution of diameters d, of parts going into service. He could then evaluate the effect of this on the distribution of stress, s. Then, the effect on part reliability could be evaluated. Finally, the effect of part reliability on system reliability would provide a <u>rational</u> and <u>quantitative</u> basis for deciding whether the parts could be used or must be rejected. This could be a valuable tool, for frequently such decisions are made on parts which cost thousands of dollars.

The above examples will provide a motivation for the study of functions of random variables as applied to structural reliability problems. This section represents a beginning and much work remains to be done in this area. However, some very interesting results and techniques have been found, and we shall proceed with a discussion of them. We shall begin with a discussion of reasons for selecting for further study certain distributions which are thought to be important to structural reliability theory.

CHAPTER 2.2

SELECTION OF DISTRIBUTIONS WHICH ARE IMPORTANT TO STRUCTURAL RELIABILITY

Introduction

We shall now proceed to discuss the reasons for the selection of certain distributions for further study. Such arguments will usually proceed on the basis of physical principles involved. Occasionally, the conclusion will be based on analysis of data, since this will serve to reinforce the decision to include the distribution.

The Normal Distribution

The normal distribution must be included in any discussion of structural reliability. Haugen (17) has developed an Algebra of Normal Functions by which structural reliability problems are solved, assuming the distributions of all the important variables to be normal. Haugen has also published data on the distribution of strengths of structural materials (18) assuming this strength to be normally distributed, and giving its mean and standard deviation. Herd (4, p. 5) states that the normal distribution (of times to failure) has been observed for shoes, clothing, furniture, most simple electronic and mechanical parts, and it can be expected for simple parts with homogeneous deterioration properties. It is commonly used in stress - strength applications.

The normal distribution may be applicable if failure occurs after an essential substance has been used up. The time to failure could be proportional to the amount of the substance in the specimen. If the amount of the substance varies among specimens according to a normal distribution, then their life-times would be normally distributed.

In the Conclusions of his PhD Dissertation, Hayes (6, p. 121) concludes that the distribution of buckling strengths of thin-walled cylinders is normally distributed. His conclusion is based on the analysis of large numbers of test results.

Smith (20, p. 35) based on his analysis of a large number of fatigue tests on small wires concludes that the normal distribution fits the fatigue strength best in the majority of cases.

In summary, we have the physical reasoning given by Herd (4), and the analyses of Hayes (6), and Smith (20) with which to conclude that the normal distribution represents the strength of many structural materials.

It is generally agreed that strengths of materials can be represented by the normal distribution, and this makes their inclusion in structural reliability work a necessity. However, many distributions important to structural reliability are markedly skewed, and therefore are not normal.

The Lognormal Distribution

Parzen (4, p. 7) discusses the lognormal distribution in relation to its applicability to fatigue life. He mentions that various probability models give rise to the lognormal distribution. These are summarized in Aitchison and Brown, (3, Ch. 3). The most important of these models is the <u>theory of proportionate effect</u> first advanced by Kapteyn in 1903. Let $X_1, \ldots, X_n$ be a sequence of random variables which represent the magnitude at successive times of, say, the length of a crack resulting from fatigue loading.

Suppose that each magnitude X_n is related to the preceeding magnitude by the relation

$$X_n - X_{n-1} = \epsilon_n X_{n-1} \tag{2.2.1}$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent random variables. Then the <u>change</u> $(X_n - X_{n-1})$ is a random proportion ϵ_n of the previous value X_{n-1}.

From (2.2.1) it follows that

$$X_n = (1 + \epsilon_n) X_{n-1} = (1 + \epsilon_n)(1 + \epsilon_{n-1}) \ldots (1 + \epsilon_2) X_1 \tag{2.2.2}$$

From (2.2.2), one sees that for large n, X_n is the product of a large number of independent factors, no one of which is dominant. The central limit theorem of probability then shows that $\log X_n$ is normally distributed, from which it follows that X_n is lognormal.

The lognormal distribution has been advanced by Freudenthal and Gumbel (20). Consider consecutive stress cycles $S_1, S_2, \ldots S_n$ applied to a specimen and let A_n be the disrupted area within the specimen. Assuming a proportionate effect,

$$A_k - A_{k-1} = S_k A_{k-1}$$

then the extent Λ_n of damage after n stress cycles is approximately lognormal for large values of n. Parzen states however, that if one is interested in the number $N(s)$ of stress cycles producing failure it appears that the probability distribution $N(s)$ is, insofar as theoretical considerations are concerned, more likely to be described by an extreme value or Weibull rather than a lognormal distribution.

Herd (3, p. 5) gives the following reasoning. The lognormal distribution applies to situations in which several independent factors influence the outcome of an event not additively but according to the magnitude of the factor and the age of the item at the time in which the factor is applied. If the effect of each impulse is directly proportional to the momentary age of the item, then the log x is nomally distributed. In other words, x is lognormal.

Aitchison and Brown (3, p. 104) cite Day(55)* who states that the results of endurance tests of many kinds (measured in terms of effective length of life of a material or piece of equipment) are frequently lognormal.

So it is seen that the lognormal distribution must be included in any study of structural reliability.

The Weibull Distribution

The principal argument in favor of the Weibull distribution is its ability to adapt to fit many shapes, depending on its parameters. This is illustrated in Figure 2.1.

Also, Parzen (4, p. 8) discusses the extreme value distribution, of which the Weibull distribution is a special case. The extreme value distribution arises in the following way.

Let $X_1, \ldots, X_n$ be a sequence of independent observations of random variable X.

Let $Y_1 = X_1$

$Y_2 =$ maximum (X_1, X_2)

$\vdots$

$Y_n =$ maximum (X_1, X_2)

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*Reference (55) given in Aitchison and Brown.

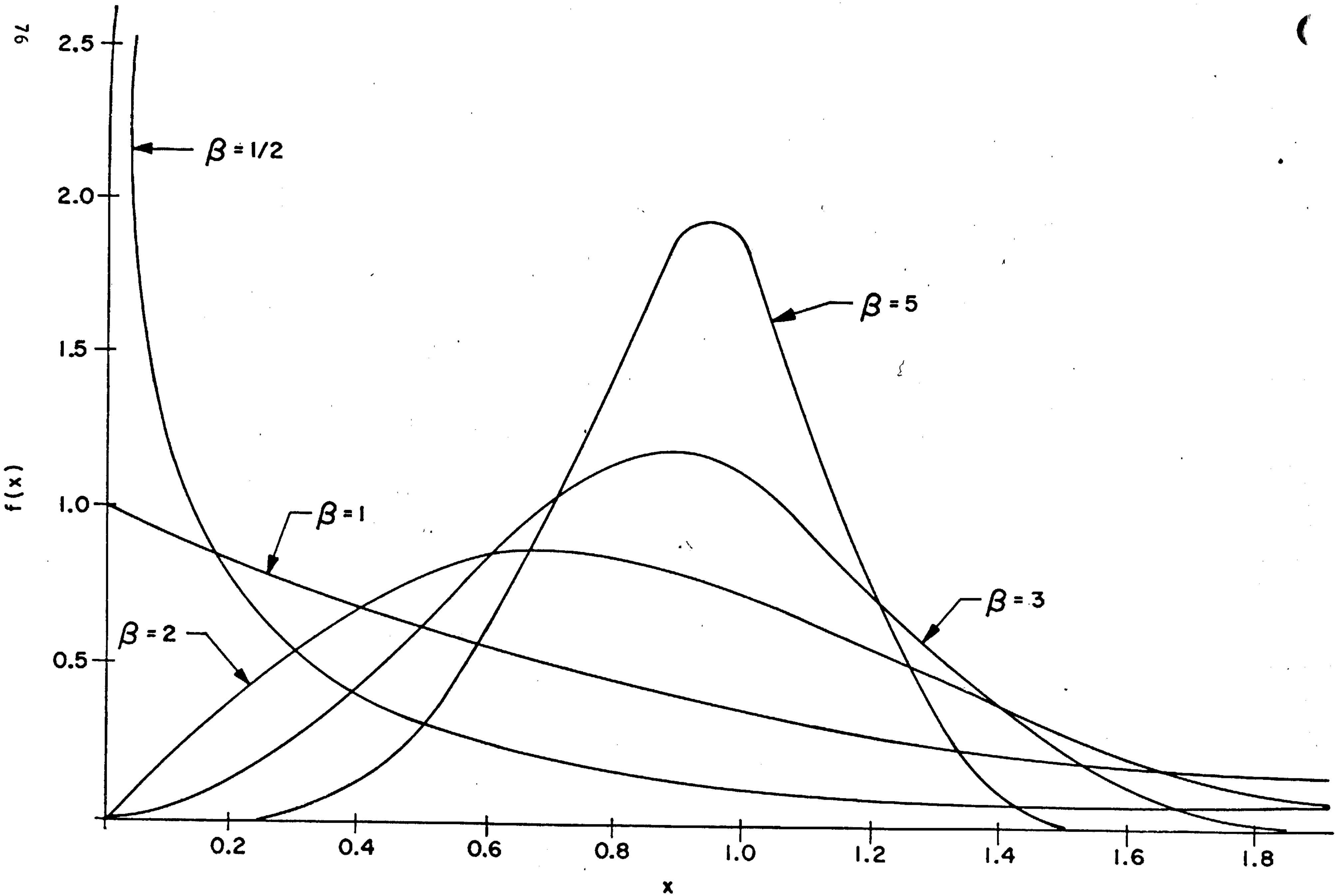

FIGURE 2.1 WEIBULL FAILURE DENSITY FUNCTION ($\eta = 1$)

and

$$Z_1 = X_1$$

$$Z_2 = \text{minimum } (X_1, X_2)$$

$$\vdots$$

$$Z_n = \text{minimum } (X_1, X_2, \ldots, X_n)$$

The random variables Y_n and Z_n are the extreme values of the observations $X_1, \ldots, X_n$. The asymptotic distribution (the distribution for large values of n) of the extreme values of Y_n and Z_n can be shown to be one of several types, depending on the distribution function of the parent $X_1, \ldots, X_n$. For a complete discussion, see Gumbel's book (21). The Weibull distribution is the Type III asymptotic distribution of extreme values, and is the asymptotic distribution of the minimum of a large number of independent observations $X_1, \ldots, X_n$ of a random variable X which cannot take values less than some lower limit γ.

If one knows the exact distribution of the independent observations $X_1, \ldots, X_n$ then one can write down an exact expression for the distribution of their values. The virtue of the theory of extreme values is that it provides an approximate method for evaluating, under a minimum of assumptions, the distribution of the maximum and minimum values in a sample.

Epstein (22) and Freudenthal and Gumbell (23) have stated the physical assumptions under which one may expect the breaking strength of a material to posess a Weibull distribution. Roughly speaking, the assumptions are that the strength of the specimen is determined by the worst flaw among the large number of flaws present in the specimen. Flaws are assumed to be distributed randomly throughout the material. The size of flaws is assumed to obey a probability distribution of type suitable for the application of the asymptotic theory of extreme values.

The Gamma Distribution

Herd (4, p. 6) states that the gamma distribution has been used to describe lifetime of electronic and mechanical systems. The gamma distribution and the Weibull distribution each have the characteristic of describing a situation in which the failure rate may be constant, increasing, or decreasing so that these two distributions can be used to describe phenomena which have widely different failure rates. Because of this ability, these two distributions have been proposed as appropriate for systems. Both appear particularly appropriate when we are dealing with a system where the deterioration mechanisms are gradually realized and controlled by redesign.

The gamma distribution is of particular interest when we consider redundant systems where each system has a constant failure rate. If we have perfect switching so that the redundant elements are inactive until called upon, i.e. in standby condition, the distribution of waiting time until all K systems fail can be represented by the gamma distribution.

Herd makes a further interesting statement to the effect that to date the multiparameter distributions are excellent in describing what happened but are of limited value in making inferences about how to improve systems. Numerous examples are present in the literature where the Weibull or gamma distribution can be used to describe data which were generated from heterogeneous populations or by heterogeneous physical processes but these lead to gross errors when extrapolated or interpolated for application to supposedly similar situations. In other words, care must be used in making reliability predictions, no matter how well past results have been described, when using the adjusted multiparameter distributions.

Birnbaum and Saunders (24) also discuss the case when times to failure follow the gamma distribution.

The gamma distribution, like the Weibull, has great flexibility. See Figure 2.2. It is included in this study for this reason, and for the physical reasons mentioned above.

The Beta Distribution

No papers which give physical reasons for the use of the beta distribution have come to the attention of the authors. Its inclusion in the study is a result of the following reasoning: An objection to, and a shortcoming of, the lognormal, Weibull, and gamma distributions is that they extend to infinity in the positive x direction. It would seem logical that a distribution which had both an upper and lower bound, such upper and lower bound being adjusted by the parameters of the distribution, might better describe certain variables such as, for example, the diameter of a rod. Such a rod in service would have, for all practical purposes, a definite upper and lower limit on its diameter. In a conversation with the authors, Dr. Jerry L. Sanders, of the Systems Engineering Department at The University of Arizona, suggested that the beta distribution had this attribute and should be considered. Therefore it was included in the present study.

Estimation of Parameters of Distributions

No theory which is developed, no matter how elegant and correct it may be, will be of any value in structural reliability if the engineer in practice cannot apply the theory in a reasonably straight forward manner to get a final number. Therefore a comment about estimation of distribution parameters is in order, although this subject will not be discussed in detail.

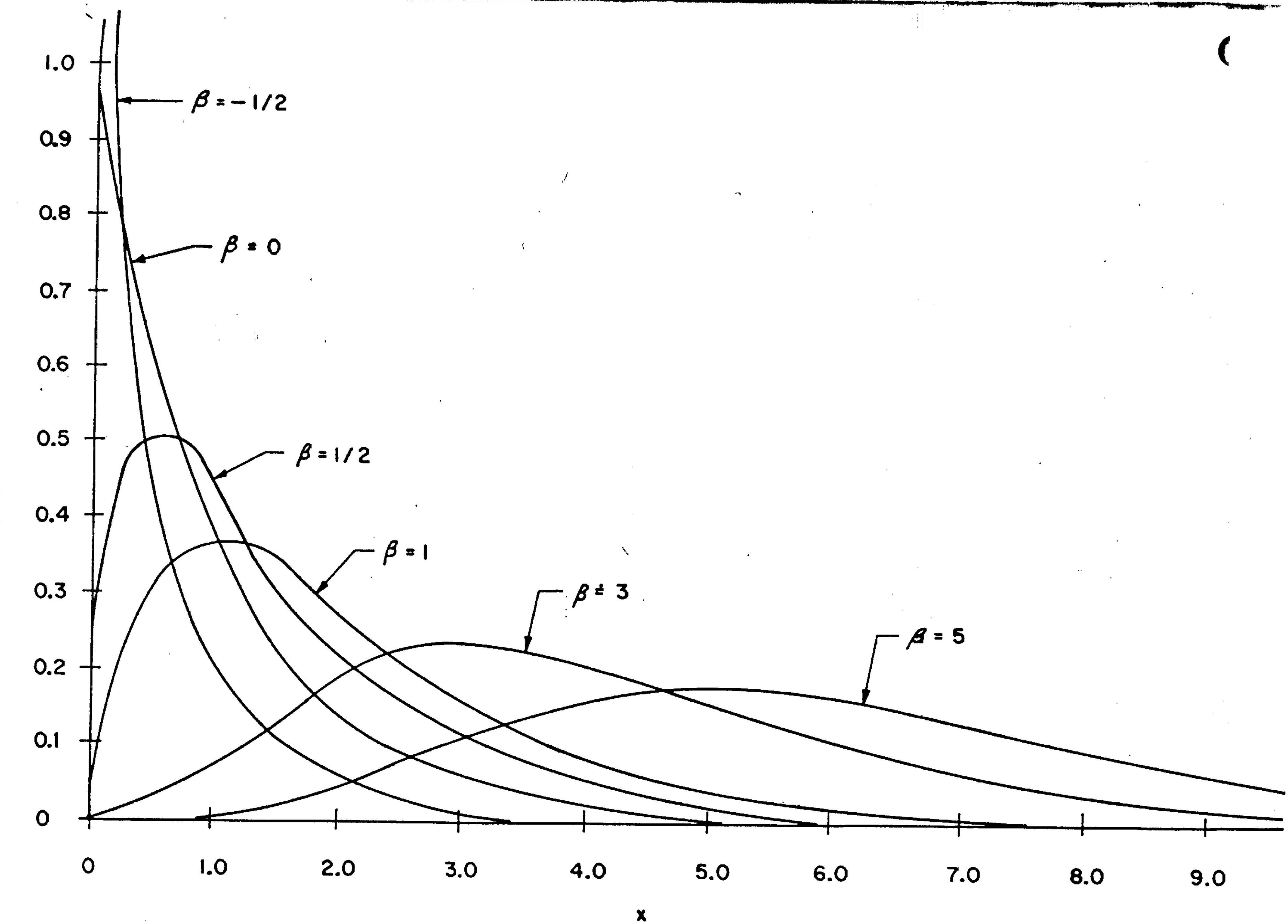

FIGURE 2.2 GAMMA FAILURE DENSITY FUNCTION $\eta = 1$

A maximum likelihood estimate for the parameters of the normal and lognormal distributions is easily obtained. See for example (25). In the case of the Weibull, gamma, and beta distributions, the situation is not quite so fortunate. The following references are suggested.

Distribution	Reference
Weibull	(26) (49)
Gamma	(26-28) (49)
Beta	(47)

The case of estimating normal and lognormal distribution parameters and the extensive tables which exist for them have caused them to be extremely popular. However, care must be taken not to use these distributions carelessly. Structural reliability results may be sensitive to the distribution which is chosen (6, p. 121), (29). One of the purposes of studies such as this is to investigate the appropriate choice of distributions based on logical or physical reasoning.

Summary

A recent summary of work in this particular area is given by Kao (49). Kao shows that the "chain model" theory of failure, where an item is assumed to consist of many sub items, the failure of the weakest one causing a failure of the item, lead either to the Weibull distribution or to Gumbel distribution (asymptotic distribution of Type I). The "rope model" whereby each item is assumed to consist of many parallel strands, and failure does not occur until all strands are broken, is shown to lead to either the gamma or the normal distribution. This paper also gives methods for estimating the parameters for these distributions, and provides an excellent summary reference for the work discussed in this Chapter. Based on the works discussed above, the following distributions have been selected for further study.

(1) Normal

(2) Lognormal

(3) Weibull

(4) Gamma

(5) Beta

The literature search and theoretical study concentrated on results for these particular distributions and on mathematical methods which would handle these distributions. Before discussing these methods and results, some necessary mathematical concepts will be discussed. This will be done in the next Chapter.

CHAPTER 2.3

MATHEMATICAL DEFINITION OF IMPORTANT DISTRIBUTIONS

The Normal p d f

The normal distribution p d f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad (2.3.1)$$

where:

$-\infty < x < \infty$

μ is the mean of the distribution, and

σ is the standard deviation

To estimate the parameters of the distribution from N observations, the following formulas are used

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N} \qquad (2.3.2)$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}} \qquad (2.3.3)$$

If there are less than 25 observations, an estimate for the standard deviation should be made by using

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N-1}} \qquad (2.3.4)$$

We shall denote the normal distribution by $N(\mu, \sigma)$.

The Lognormal p d f

The lognormal p d f is defined by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2} \log\left(x - \mu\right)^2\right] \qquad (2.3.5)$$

where: $0 \leq x < \infty$

Aitchison and Brown (3) is recommended for an excellent discussion of the lognormal distribution.

The parameters are the mean μ and the standard deviation σ. These can be estimated from N observations by

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \log x_i \qquad (2.3.6)$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (\log x_i - \mu)^2}{N}} \qquad (2.3.7)$$

If N is less than 25, the following equation should be used to estimate the standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (\log x_i - \mu)^2}{N-1}} \qquad (2.3.8)$$

The lognormal distribution will be denoted by $L(\mu, \sigma)$.

The Weibull p d f

The Weibull p d f is defined by

$$f(x) = \frac{\beta}{\eta} \left(\frac{x-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}} \qquad (2.3.9)$$

where: $\gamma \leq x < \infty$
γ is the location parameter ≥ 0
β is the shape parameter ≥ 0
η is the scale parameter > 0

The mean of the Weibull distribution is given by (31),

$$\mu = \text{mean} = \gamma + \eta \Gamma \left(1 + \frac{1}{\beta}\right), \tag{2.3.10}$$

and the variance of the Weibull distribution is given by

$$\sigma^2 = \eta^2 \left[\Gamma \left(1 + \frac{2}{\beta}\right) - \Gamma^2 \left(1 + \frac{1}{\beta}\right)\right] \tag{2.3.11}$$

where $\Gamma(\xi)$ is the gamma function of ξ defined by

$$\Gamma(\xi + 1) = \int_0^\infty e^{-x} x^\xi \, dx \tag{2.3.12}$$

Tables of the gamma function can be found in Ref. (32).

The Weibull distribution will be denoted by $W(\gamma, \beta, \eta)$.

The Gamma p d f

$$f(x) = \frac{e^{-x/\eta} \, x^\beta}{\Gamma(\beta + 1) \, \eta^{\beta + 1}} \tag{2.3.13}$$

where: $0 \leq x < \infty$
η is the scale parameter ≥ 0
β is the shape parameter $\geq (-1)$

The mean of the gamma distribution is given by (30, p. 93)

$$\mu = (\beta + 1) \, \eta \tag{2.3.14}$$

and the variance by

$$\sigma^2 = (\beta + 1) \, \eta^2 \tag{2.3.15}$$

We shall use $G(\beta, \eta)$ to denote the gamma distribution.

The Beta p d f

There are two kinds of beta distributions, defined as follows. The beta distribution of the first kind is given by

$$f(x) = \frac{1}{B(a,\ b)} \ x^{a-1} \ (1 - x)^{b-1} \qquad (2.3.16)$$

$$\text{where: } \begin{array}{l} o < x < 1 \\ a > o \\ b > o \end{array}$$

$$B(a,\ b) = \frac{\Gamma(a)\ \Gamma\ (b)}{\Gamma\ (a + b)} \qquad (2.3.17)$$

The beta distribution of the second kind is given by

$$f(x) = \frac{x^{a-1}}{B(a,\ b)(1 + x)^{a + b}} \qquad (2.3.18)$$

These two p d f's will be denoted by $B_1(a,\ b)$ for the beta distribution of the first kind and by $B_2(a,\ b)$ for the beta distribution of the second kind.

Summary

The mathematical formulas of the p d f's selected for future study have been given. Now we shall proceed to a discussion of mathematical concepts and definitions.

CHAPTER 2.4

MATHEMATICAL CONCEPTS AND DEFINITIONS

Introduction

In this Chapter some essential concepts of mathematical statistics will be discussed. An excellent book by Hogg and Craig (30) gives a more complete discussion. The basic concepts presented here will be essential to understanding the mathematical methods used to study functions of random variables discussed in Chapter 2.5.

The Probability Density Function

The probability density function for a continuous random variable can be defined in the following way (30, p. 17). Let the one - dimensional set A be such that the Riemann integral

$$\int_A f(x)dx = 1$$

and the following conditions are met:

1. $f(x) > 0$ for all x in A.

2. $f(x)$ has at most a finite number of discontinuities in every finite interval that is a subset of A.

Then if the probability of a set a which is a subset of A is given by

$$P(a) = Pr(X \in a) = \int_a f(x)dx$$

X is said to be a random variable of the continuous type, and $f(x)$ is said to be the probability density function of X. The abbreviation p d f will be used for the probability density function.

The Cumulative Distribution Function

The Cumulative Distribution Function (c d f), denoted by $F(x)$, can be defined as

$$F(x) = Pr\ (X \leq x)$$

and is related to the p d f by

$$F(x) = \int_{-\infty}^{x} f(x)dx \qquad (2.4.1)$$

for the continous type of random variable.

Mathematical Expectation

A useful concept in problems involving distributions of random variables is the mathematical expectation, (30, p. 34). If a continuous random variable has p d f $f(x)$, and $u(X)$ is a function of X such that

$$\int_{\infty}^{\infty} u(x)\ f(x)\ dx \qquad (2.4.2)$$

exists, then the integral (2.4.2) is defined to be the mathematical expectation of $u(X)$. This is denoted by $E[u(X)]$.

The extension of the definition to functions of more than one variable follows the expected pattern

$$E[u(X_1,\ \ldots,\ X_n)]$$

$$= \int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} u(x_1,\ \ldots,\ x_n)\ f(x_1,\ \ldots,\ x_n)dx_1 \ldots dx_n \qquad (2.4.3)$$

Special Mathematical Expectations

Letting $u(x) = X$ results in

$$E(X) = \int_{\infty}^{\infty} xf(x)\ dx \qquad (2.4.4)$$

which is the mean value of X. Adopting the symbol

$$E(X) = \mu$$

we can further define

$$E\left[(X-\mu)^2\right] = \int_{\infty}^{\infty} (x-\mu)^2\ f(x)\ dx \qquad (2.4.5)$$

as the variance of X, denoted by σ^2.

A third special mathematical expectation is called the moment generating function, which is given by

$$E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \qquad (2.4.6)$$

We shall adopt the symbol M(t) for the moment generating function, that is

$$M(t) = E\left(e^{tx}\right)$$

The moment-generating function does not always exist, because there are some distributions which do not have a moment-generating function. However, when it does exist, a <u>most important</u> statement can be made. The moment-generating function is unique, that is, if two random variables have the same moment-generating function, then they have identically the same distribution (30, p. 40).

The moment-generating function gets its name from the following properties : Consider:

$$M(t) = E\left(e^{tx}\right)$$

Then

$M'(t)\Big|_{t=0}$ is the first moment of the distribution, or the mean

$M''(t)\Big|_{t=0}$ is the second moment of the distribution

$M^{n}(t)\Big|_{t=0}$ is the n^{th} moment of the distribution

Therefore, any moment of the distribution can be generated by taking the desired derivative and setting t equal to zero. Note also that

$$\begin{aligned}\sigma^2 &= M''(o) - \mu \\ &= M''(o) - M'(o)\end{aligned} \qquad (2.4.7)$$

Marginal Distributions (30, p. 54)

If $f(x_1, x_2)$ is the joint p d f of two random variables X_1 and X_2, the event $a < X_1 < b$, where $a < b$, is possible when and only when the event a $a < X_1 < b;\ -\infty < X_2 < \infty$ occurs. This can be written as the following probability statement:

$$Pr(a < X_1 < b;\ -\infty < X_2 < \infty) = \int_a^b \int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1$$

This is the integral of the joint p d f over the limits of the probability event being considered.

Consider the integral

$$\int_{\infty}^{\infty} f(x_1, x_2)dx_2 \tag{2.4.8}$$

this is a function of x_1 alone, and is <u>defined</u> to be the marginal p d f of X_1, denoted $f_1(x_1)$. Similarly,

$$f_2(x_2) = \int_{\infty}^{\infty} f(x_1, x_2)dx_1 \tag{2.4.9}$$

The concept of marginal p d f will be essential later when the change of variable technique is discussed.

Stochastic Independence

Most of the results for functions of random variables are given for random variables which are stochastically independent (often called merely, "independent"). Stochastic independence can be defined as follows (30, p. 68):

> Let the random variables X_1 and X_2 have the joint p d f $f(x_1, x_2)$ and the marginal probability density functions $f_1(x_1)$ and $f_2(x_2)$ respectively. The random variables X_1 and X_2 are said to be stochastically independent if, and only if, $f(x_1, x_2) = f_1(x_1)\ f_2(x_2)$.

The change of variable technique to be discussed later will utilize this definition. In general, the formation of the joint p d f of n independent random variables $X_1, \ldots, X_n$ can be written as

$$f(x_1, \ldots, x_n) = f_1(x_1) \times \ldots \times f_n(x_n) \tag{2.4.10}$$

Summary

The following concepts from mathematical statistics have been defined and discussed:

1. Probability Density Function
2. Cumulative Distribution Function

3. Mathematical Expectation

4. Marginal Distributions

5. Stochastic Independence

These definitions will be applied in the next chapter, where we shall discuss in detail some of the mathematical techniques used in the study of functions of random variables.

CHAPTER 2.5

MATHEMATICAL METHODS USED IN THE STUDY OF FUNCTIONS OF RANDOM VARIABLES

Introduction

In this chapter several methods for treating problems in functions of random variables are discussed. It is not possible to say, in general, that any one method is better than another. Depending on the distributions being studied, and the functions of these distributions which are desired, it may be that a particular method will be superior. It must also be said that not all of the functions which are desired for structural application can be obtained in closed form. However, the following techniques will usually yield results, even if numerical methods must be used. We shall now summarize eight methods, and give an example of the use of each one.

The Algebra of Normal Functions Method

Haugen (17) has presented a method, which he calls The Algebra of Normal Functions. In this method, which is primarily intended for use in structural reliability design, all variables are assumed normally distributed. It is also assumed that the results for the sum, difference, product, and quotient of two normal random variables are normally distributed*. Under these assumptions, a completely closed algebra can be developed; this is summarized as follows:

<u>Sum</u> $Z = X + Y$

where: X is $N(\mu_x, \sigma_x)$
Y is $N(\mu_y, \sigma_y)$

Then
$$\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2} \qquad (2.5.1)$$
$$\mu_z = \mu_x + \mu_y$$

*This is valid exactly only for the sum and difference of two normally distributed random variables.

Difference $Z = X - Y$

where X is $N(\mu_x, \sigma_x)$

Y is $N(\mu_y, \sigma_y)$

Then
$$\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2} \tag{2.5.2}$$
$$\mu_z = \mu_x - \mu_y$$

Product $Z = X \cdot Y$

where X is $N(\mu_x, \sigma_x)$

Y is $N(\mu_y, \sigma_y)$

Then
$$\sigma_z = \sqrt{\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \sigma_x^2\sigma_y^2} \tag{2.5.3}$$
$$\mu_z = \mu_x \cdot \mu_y \tag{2.5.4}$$

Quotient $Z = X/Y$

where X is $N(\mu_x, \sigma_x)$

Y is $N(\mu_y, \sigma_y)$

Then
$$\sigma_z = \frac{1}{\mu_y}\sqrt{\frac{\mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2}{\mu_y^2 + \sigma_y^2}} \tag{2.5.5}$$
$$\mu_z = \mu_x/\mu_y \tag{2.5.6}$$

Smith (19, p. 40) has discussed the limits of application of equation (2.5.5). The quotient of two normal random variables is approximately normal when $\mu_y/\sigma_y > 4$, and equation (2.5.5) is a good approximation. An equation by Marsaglia (33) is more accurate when $\mu_y/\sigma_y < 4$. Further discussion of the quotient of two normal variables will follow later in Chapter 2.6.

This work by Haugen represent a major step forward in the attempt to incorporate structural reliability at the design level and does allow the reliability approach to be taken. It cannot be denied however that some variables which are significant in structural applications are not normally distributed. Therefore we shall discuss some additional techniques which can be applied to variables which are not normal.

The Change of Variable Method

The discussion here will follow that of Hogg and Craig (30, Ch. 4); however, this technique is quite common, and can be found in many books on mathematical statistics. The technique proceeds as follows:

Suppose we have $X_1, \ldots, X_n$ stochastically independent random variables each with p d f $f(x)$. Then the joint p d f of $X_1, \ldots, X_n$ is given by (for independent random variables)

$$f(x_1, \ldots, x_n) = f(x_1)\, f(x_2) \ldots f(x_n) \tag{2.5.7}$$

Now if we desire to obtain the p d f of a function $Y_1 = u_1(X_1, \ldots, X_n)$ a somewhat roundabout method must be used. First, we must form n functions which define a one-to-one transformation,

$$\begin{aligned} y_1 &= u_1(x_1, \ldots, x_n) \\ y_2 &= u_2(x_1, \ldots, x_n) \\ &\ldots\ldots\ldots\ldots \\ y_n &= u_n(x_1, \ldots, x_n) \end{aligned} \tag{2.5.8}$$

It is important to note here that we must have n transformation functions, even though we may only desire to know one particular function, say $y_1 = u_1(x_1, \ldots, x_n)$.

Now define n inverse transformation functions of the transformation given by equation (2.5.8)

$$\begin{aligned} x_1 &= w_1(y_1, \ldots, y_n) \\ x_2 &= w_2(y_1, \ldots, y_n) \\ &\ldots\ldots\ldots\ldots \\ x_n &= w_n(y_1, \ldots, y_n) \end{aligned} \tag{2.5.9}$$

Then it follows from work in analysis regarding change of variables in an integral (34, p. 243), that the joint p d f $g(y_1, \ldots y_n)$ is given by,

$$g(y_1, \ldots, y_n) = |J|\, \varphi \left[w_1(y_1, \ldots, y_n), \ldots, w_n(y_1, \ldots, y_n)\right] \qquad (2.5.10)^{*}$$

* $|J|$ = absolute value of J .

where:

J is the Jacobian of the transformation and is defined to be

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

Now that the joint p d f of $Y_1, \ldots Y_n$ is known, we can find our desired function $Y_1 = u_1(X_1, \ldots X_n)$ by finding the marginal p d f

$$g_1(y_1) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} g(y_1, \ldots, y_n)\, dy_2 \ldots dy_n \qquad (2.5.11)$$

We shall now illustrate this technique by a simple example. For purposes of illustration, we will use a simple example - one whose results may not be too interesting for structural applications, but which serves to illustrate the method.

Example (30, p. 123) Let $Y_1 = (X_1 - X_2)$, where X_1 and X_2 are independent random variables, each being χ^2 (2). * Find the p d f of Y_1.

The chi-square distribution with r degrees of freedom is

$$f(x) = \frac{1}{\Gamma\left(r/2\right) 2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 < x < \infty \qquad (2.5.12)$$

$$= 0 \qquad \text{elsewhere}$$

(a) The joint p d f of X_1 and X_2 is

$$f(x_1)\, f(x_2) = \exp\left(-\frac{x_1 + x_2}{2}\right) \qquad (2.5.13)$$

$$0 < x_1 < \infty,\ 0 < x_2 < \infty$$

(b) Arbitrarily, let $Y_2 = X_2$ since we need two one-to-one transformations. Then the equations of transformation are

$$y_1 = (x_1 - x_2) \qquad (2.5.14)$$
$$y_2 = x_2$$

* $\chi^2(r)$ will symbolize the chi-square distribution with r degrees of freedom.

(c) The inverse transformation is

$$x_1 = 2y_1 + y_2 \qquad -2y_1 < y_2$$

$$x_2 = y_2 \qquad \text{and} \quad 0 < y_2; \; -\infty < y_1 < \infty \qquad (2.5.15)$$

(d) The Jacobian of the inverse transformation is

$$J = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

(e) The joint p d f of Y_1 and Y_2 is

$$g(y_1, y_2) = \tfrac{1}{2} e^{-y_1 - y_2}$$

(f) The marginal p d f of Y_1 is

$$g_1(y_1) = \int_{-2y_1}^{\infty} \tfrac{1}{2} e^{-y_1 - y_2} \, dy_2 = \tfrac{1}{2} e^{y_1} \qquad -\infty < y_1 < 0$$

$$\int_{0}^{\infty} \tfrac{1}{2} e^{-y_1 - y_2} \, dy_2 = \tfrac{1}{2} e^{-y_1} \qquad 0 < y_1 < \infty$$

$$g_1(y_1) = \tfrac{1}{2} e^{-|y_1|} \qquad -\infty < y_1 < \infty \qquad (2.5.16)$$

The extension of this technique to functions of several variables follows the same pattern. One disadvantage of this technique is the requirement for generating as many equations of transformation as there are random variables involved. It is frequently possible to get around this requirement by using the moment generating function method, which will be discussed next.

The Moment Generating Function Method

In Equation (2.4.6) the moment generating function was defined as

$$E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Now it can be shown (30, p. 135) that we can compute $E\left[\exp\left(t\, u_1(x_1, \ldots x_n)\right)\right]$ and have the value of $E(e^{tY_1})$ where $Y_1 = u_1(X_1, \ldots X_n)$. Since the moment generating function is unique, then if the moment generating function of Y_1 is seen to be that of a certain kind of distribution, then Y_1 has that distribution.

Example (30, p. 138): Let X_1 and X_2 be independent random variables with normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$, respectively. Let $Y = X_1 - X_2$; find $g(y)$, the p d f of Y. The moment generating function of Y is

$$M(t) = E(e^{t(x_1 - x_2)})$$

$$= E\left(e^{tx_1}\right) E\left(e^{-tx_2}\right) \qquad (2.5.17)$$

The form of the moment generating function for the normal distribution is known to be (30, p. 97)

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \qquad (2.5.18)$$

Then

$$E(e^{tx_1}) = \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right)$$

$$E(e^{-tx_2}) = \exp\left(-\mu_2 + \frac{\sigma_2^2 t^2}{2}\right)$$

So that $M(t)$ from equation (2.5.17) is

$$M(t) = \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) \exp\left(-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)$$

$$= \exp\left(\left(\mu_1 - \mu_2\right) t + \frac{\sigma_1^2 + \sigma_2^2}{2} t^2\right) \qquad (2.5.19)$$

From equation (2.5.19) it is seen that $g(y)$, the p d f of Y is $N(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$.

The moment generating function method is quite easy when it works, but it may not always work, because some functions do not possess moment generating functions; moreover, the result of applying the moment generating function method may not be recognizable. We are now going to proceed to a discussion of a method, the Fourier transform method, which will be especially suited to problems involving sums of random variables.

The Fourier Transform Method (35), (36), (38)

This method, utilizing the Fourier transform, convolution, and inversion is a very powerful one in the study of sums of random variables, especially since numerical techniques exist for solution, Ref. (35).

We have, from Ref. (37), the following theorem:

If X obeys a law $\int_{\infty}^{\infty} f_1(x)dx = 1$, and Y obeys a law $\int_{\infty}^{\infty} f_2(y)dy = 1$ then the sum $Z = X + Y$ will obey the law

$$\int_{\infty}^{\infty} \int_{\infty}^{\infty} f_1(z - y)\, f_2(y)dydz = 1$$

That is, the p d f of z will be

$$g(z) = \int_{\infty}^{\infty} f_1(z-y)\, f_2(y)\, dy \tag{2.5.20}$$

But this is recognized to be the inverse of the Fourier convolution (36, Sec. 3.33) as follows:

The Fourier transforms of $f_1(x)$ and $f_2(y)$ are

$$F_1(u) = \int_{\infty}^{\infty} f_1(x)e^{-iux}dx$$

$$F_2(u) = \int_{\infty}^{\infty} f_2(y)e^{-iuy}dy \tag{2.5.21}$$

Then

$$F^{-1}\left[F_1(u)\quad F_2(u)\right] = g(z)$$

$$= \int_{\infty}^{\infty} f_1\,(z-y)\, f_2(y)dy \tag{2.5.22}$$

where F^{-1} is the inverse Fourier transform.

This says that we can find the p d f $g(z)$ of the sum of two random variables X and Y such that $Z = X + Y$ by finding the Fourier transforms of X and Y, multiplying them together, and inverting the result.

The inverse of the Fourier transform is defined to be

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} F(u)du \qquad (2.5.23)$$

Tables exist for the transform and its inverse, see (35, p. 122).

<u>Example</u>: (36, Ex. 3.3-2). Let

$$\begin{aligned} f_1(x) &= 1 \qquad 0 < x < 1 \\ &= 0 \qquad \text{elsewhere} \\ f_2(y) &= e^{-y} \qquad y < 0 \end{aligned}$$

It is desired to find g(z), the p.d.f. of the random variable $Z = X + Y$. Then we have*

$$F_1(u) = \frac{1}{iu} - \frac{e^{-iu}}{iu}$$

$$F_2(u) = \frac{1}{iu + 1}$$

So that

$$F_1(u) \cdot F_2(u) = \left[\frac{1}{iu(iu + 1)} - \frac{e^{-iu}}{iu(iu + 1)}\right] = F(u)$$

and

$$\begin{aligned} g(z) &= F^{-1}(u) \\ &= 1 - e^{-z} \qquad z < 1 \\ &= (e - 1)e^{-z} \qquad 1 \leq z \leq \infty \end{aligned}$$

This illustrates the utility of the Fourier transform method in solving problems involving the sum of two random variables. A similar method, the Mellin transform method is especially suited to solving problems involving products and quotients of random variables.

The Mellin Transform, Convolution, and Inversion Method (7), (8), (35), (39)

The Mellin transforms and convolutions are very useful in finding the

*Note that in this case the Fourier transform can be found by looking up the Laplace transform and replacing s by iu. A similar process provides the inverse.

products and quotients of independent random variables. This is a result of their following properties (7, p. 7):

1. The p d f of n independent random variables may be expressed as a Mellin convolution.

2. The Mellin convolution yielding the p d f of a product of n independent random variables can be obtained from its Mellin transform by means of an inversion formula.

3. The transform of the Mellin convolution yielding the p d f of a product of n independent random variables is the product of the Mellin transforms of the p d f's of the component random variables.

The Mellin transform can be defined as (7, p. 8)

$$M(f(x)) = E\left[X^{s-1}\right] = \int_{-\infty}^{\infty} x^{s-1} f(x)dx \qquad (2.5.24)$$

Note that the Mellin transform is defined only for $x \geq 0$. However, Epstein (8) and Springer and Thompson (7, p. 10) discuss systems whereby the transform can be applied over the range $-\infty \leq x \leq \infty$. This technique involves breaking the function into positive and negative parts, and redefining the partial functions in such a way that the Mellin transform method can be applied. The details will not be presented here. The presentation in this report will be restricted to the range $x \geq 0$, with the understanding that the method can be extended.

The inverse formula is given by

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M(f(x))\, ds \qquad (2.5.25)$$

Tables for the the Mellin transform and inverse are available, (8), (35, p. 255), (43).

The Mellin convolution of two functions $f_1(x)$ and $f_2(x)$, $0 \leq x < \infty$ is defined as

$$g(x) = \int_0^{\infty} \left(\frac{1}{y}\right) f_2\left(\frac{x}{y}\right) f_1(y)dy \qquad (2.5.26)$$

But this is exactly the p d f of the product $h(x)$ where $X = X_1X_2$ of two independent random variables with p d f's $f_1(x_1)$ and $f_2(x_2)$, (37).

One now has

$$M(h(x)) = E\left[(x_1x_2)^{s-1}\right]$$
$$= \left[E\ (x_1^{s-1})\right]\left[E\ (x_2^{s-1})\right]$$
$$= M(f_1(x_1))\ \ M(f_2(x_2)) \qquad (2.5.27)$$

Equation (2.5.27) says that the p d f of a product of two independent random variables X_1 and X_2 is the Mellin convolution whose transform is the product of the Mellin transforms $f_1(x_1)$ and $f_2(x_2)$.

By successive application of the above scheme, we can find the product of n independent random variables $X_i \geq 0$ with p d f's $f_i(x_i)$, i=1, 2, ..., n by the use of

$$M(h_n(x)) = \prod_{i=1}^{n} M(f_i(x_i)) \qquad (2.5.28)$$

The p d f of the desired result, $h_n(x)$, can be obtained by inverting $M(h_n(x))$ by use of tables, or by applying the definition of the inverse

$$h_n(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} \prod_{i=1}^{n} M(f_i(x_i)) ds \qquad (2.5.29)$$

This integral can, in most cases, be evaluated by use of the theory of residues, see, for example, (40, Ch. 12).

The Mellin transform method can also be used to find the distributions of quotients of independent random variables in the following way. Consider the Mellin transform of the function $g(x) = x^{\alpha(s-1)}$, then

$$E\left[X^{\alpha(s-1)}\right] = \int_0^\infty x^{\alpha(s-1)} f(x)dx \qquad (2.5.30)$$
$$= M\left[f(x)\ \middle|\ \alpha s-\alpha+1\right] *$$

If $\alpha = -1$, then

$$E\left[(X^{-1})^{s-1}\right] = M(f(x)\ |-s+2) \qquad (2.5.31)$$

---

*The symbol $M\left[f(x)\ |\ \alpha s - \alpha + 1\right]$ will mean that the expression is $M(f(x))$ with s replaced by $\alpha s - \alpha + 1$. In general, $M\left[f(x)|\ \xi\right]$ will mean $M(f(x))$ with s replaced by ξ.

That is, the Mellin transform of the p d f of the reciprocal of a random variable, $1/X$, is the Mellin transform of the X, with the parameter s replaced by $-s+2$.

Then one can determine the p d f $q(z)$ of the quotient of two random variables X_1 and X_2 with p d f's $f_1(x_1)$ and $f_2(x_2)$ by

$$M(q(z)) = M(f_1(x_1)) \cdot \boxed{M \quad f_2(x_2) \Big| -s+2} \tag{2.5.32}$$

Then $q(z)$ will follow from the inversion

$$q(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M(f_1(x_1)) \cdot \boxed{M \quad f_2(x_2) \Big| -s+2} \, ds \tag{2.5.33}$$

Example: Let the random variable X have the mononomial p d f

$$\begin{aligned} f(x) &= (\alpha+1)\, x^{\alpha} && 0 \leq x \leq 1, \quad \alpha \text{ real} \\ &= 0 && \text{elsewhere} \end{aligned}$$

The Mellin transform of the product $Y = \prod_{i=1}^{n} X_i$ is, using elementary integration and Equations (2.5.24) and (2.5.28).

$$M(h(y)) = \left[\frac{(\alpha+1)}{s+\alpha}\right]^n$$

The inverse is

$$h(y) = \frac{(\alpha+1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\,(s+\alpha)^{-n}\, ds$$

which, by application of the residue theorem, becomes

$$\begin{aligned} h(y) &= \frac{(\alpha+1)^n}{(n-1)!}\, y^{\alpha}\left(\ln \frac{1}{y}\right)^{n-1} && 0 \leq y \leq 1 \\ &= 0 && \text{elsewhere} \end{aligned}$$

The power and utility of the Mellin transform method for finding the distributions of products and quotients of random variables can now be seen. Some results of the application of this method will be discussed in Chapter 2.6. We will now discuss a method which is related to the moment generating function method, namely, the characteristic function method.

The Characteristic Function Method (41, p. 240; p. 247)

In Chapter 2.4 it was mentioned that a moment generating function did not always exist for a given distribution. A closely allied function which will always exist is the characteristic function, which is defined as

$$\varphi\,(t) = \int_{-\infty}^{\infty} e^{itz}\, f(z)\, dz \qquad (2.5.34)$$

As in the case of the moment generating function, a distribution is completely determined by its characteristic function.

Also, if the characteristic function is known, the distribution function can be found from the relation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi\,(t)\, e^{-itx}\, dx \qquad (2.5.35)$$

This method will not be developed to a great extent at this time; a simple example to illustrate its application will be presented.

<u>Example (41, p. 275)</u>: If we have n independent random variables $X_1, \ldots, X_n$ whose characteristic functions are $\varphi_1(t), \ldots, \varphi_1(t)$, the product

$$\varphi\,(t) = \varphi_1(t)\; \varphi_2(t)\; \ldots\; \varphi_n(t)$$

is the characteristic function of the sum

$$S = X_1 + X_2 + \ldots + X_n$$

This can be seen by considering

$$\varphi\,(t) = E\left[e^{ist}\right]$$

$$= E\left[e^{ix_1t} \cdot e^{ix_2t} \ldots e^{ix_nt}\right]$$

but since the random variables are independent, the expectation of the product is equal to the product of the expectations, therefore

$$\varphi(t) = \varphi_1(t)\,\varphi_2(t)\,\ldots\,\varphi_n(t)$$

Consider now n independent normally distributed random variables with means $= 0$ and standard deviation $\sigma_1, \sigma_2, \ldots \sigma_n$. Their characteristic functions are (41, p. 244)

$$\varphi_k(t) = e^{-\frac{\sigma_k^2 t^2}{2}} \qquad k = 1, 2, \ldots n$$

and the characteristic function of their sum

$$S = X_1 + X_2 + \ldots + X_n$$

will be

$$\varphi(t) = e^{-\frac{\sigma^2 t^2}{2}}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots \sigma_n^2$

Therefore, S is normally distributed with mean 0 and standard deviation

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}$$

The Cumulative Distribution Function Method

This method is quite simple to apply, and is actually the one-dimensional case of the change of variable technique. However, it is presented separately because it provides a handy tool for finding functions of random variables such as the one presented in the following example (30, p. 95).

<u>Example:</u> Let X have a gamma distribution with $\beta = r/2 - 1$ where r is a positive integer and $\beta > 0$. If $Y = 2X/\eta$, what is the p d f of Y?

The cumulative distribution function of Y is

$$G(y) = P_r(Y \leq y) = Pr\left(X \leq \frac{\eta y}{2}\right)$$

For $y > 0$,

If (r/2 - 1) is substituted for β in Equation (2.3.13), then

$$G(y) = \int_0^{\eta y/2} \frac{1}{\Gamma\left(\frac{r}{2}\right)\eta^{r/2}} x^{(r/2-1)} e^{-x/\eta}\, dx$$

The p d f of Y is

$$g(y) = G'(y)$$

$$= \frac{\eta/2}{\Gamma\left(\frac{r}{2}\right)\eta^{r/2}} (\eta y/2)^{(r/2-1)} e^{-y/2}$$

$$= \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} y^{(r/2-1)} e^{-y/2}$$

or, Y is $\chi^2(r)$

The Monte Carlo Method

The Monte Carlo method enables the determination of the distribution of the dependent variable, given the distribution of the variables it is a function of, by means of computer simulation. Many of the functions of random variables for stress and strength, as well as other important parameters, can be synthesized and evaluated via Monte Carlo simulation. The Monte Carlo technique is a popular and successful one. The following description covers the determination of the distribution of the dependent variable from the distribution of several independent variables:

1. Divide the distribution of each independent variable or factor into an optimum number of intervals.

2. Calculate the centroid C_{ij} for all the intervals in each distribution.

3. Determine the probability of occurrence of the centroid of each interval, P_{ij}, which is the percentage of the total area under the ith distribution contained in the jth interval.

4. Enter all the pairs of numbers, C_{ij} and P_{ij}, into a digital computer, along with a random number generating program for associating a particular digit or digits, in the random number, with a particular variable and a particular pair of C_{ij} and P_{ij} values of that variable.

5. Generate a random number and therefrom identify a complete set of C_{ij} values for all variables identified with it, together with their

associated probabilities, P_{ij} . Continue until over 250 and preferably up to 2000 such sets are obtained.

6. Calculate the response (strength, stress, or reliability) from each set of the randomly selected variables.

7. For each response value determine the product of the interval probabilities, π_e , associated with the C_{ij} used in calculating the response.

8. Determine a suitable number of intervals, k , for the calculated response values. Group the values of the response lying in each interval and their probabilities.

9. Calculate the centroidal response, r_{ck} , for each interval. These are the abscissas of the response histogram.

10. Determine the probability of occurrence associated with each r_{ck} , or P_{ck} , by summing up the probabilities associated with the response per internal width. These are the ordinates of the response relative frequency histogram. They are converted to frequencies by multiplying by a number of trials. If not multiplied, then we have a relative frequency histogram.

11. Plot the centroidal response values, r_{ck} , and the associated probabilities, P_{ck} . This will yield a response relative frequency histogram which is properly weighted relative to the probability density functions of the initial variables involved.

12. A distribution curve may now be fitted by statistical regression analysis which is the distribution of the dependent variable.

The Monte Carlo technique may be applied to any combination of distributions. The accuracy increases as the number of intervals chosen for the probability density function of the variables is increased, and as the number of the calculated response values is increased.

Summary

In this chapter eight methods for attacking problems in the study of functions of random variables have been presented. These are:

1. The Algebra of Normal Functions Method

2. The Change of Variable Method

3. The Moment Generating Function Method

4. The Fourier Transform, Convolution, and Inversion Method

5. The Mellin Transform, Convolution, and Inversion Method

6. The Characteristic Function Method

7. The Cumulative Distribution Function Method

8. The Monte Carlo Method

It cannot be stated categorically that one method is to be favored over all others. The Algebra of Normal Functions method has the advantages of being easy and straightforward, and the disadvantages of treating only normal functions and of being inexact except for the sum and difference. The change of variable method, moment generating function method, characteristic function method, and cumulative distribution function method may yield good results in some cases but may become involved and cumbersome in other cases.

The two methods, Mellin and Fourier transform, are restricted respectively to the product and quotient for the Mellin transform method and to the sum for the Fourier transform method. These methods are very powerful for these particular cases, especially when it is noted they can be handled by numerical techniques when they fail to yield closed solutions.

The Monte Carlo method will always give an answer, even for complex functions of non-identically distributed random variables. This, in view of the present state of the art of the other methods discussed, makes it a very powerful and valuable tool. It does, however, require the use of a digital computer and can thus be a costly method.

Some applications of these methods, and some interesting results which are thought to be of importance in the study of functions of random variables for structural applications are mentioned in the next chapter.

CHAPTER 2.6

SOME SIGNIFICANT RESULTS FOR STRUCTURAL APPLICATIONS

Introduction

Some interesting results have come to light which promise to be of importance to structural reliability theory. These will be discussed below. These results, which had not come to the authors' attention before the start of the present study, represent a first step into areas which have not been heretofore utilized in solving structural reliability problems.

We will begin by discussing some results for the normal function, then functions of other distributions will be taken up.

Ratio of Normal Variables

Haugen's Algebra of Normal Functions (17) assumes that the ratio of two normal variables is normal. The validity of this assumption was discussed briefly in Chapter 2.5. It is known that the ratio of two normal variables is, in fact, not normal.

For the ratio of two standard normal variables, N(0, 1), Epstein (8, p. 377) shows, by the Mellin transform technique, that

$$h(y) = \frac{1}{\pi} \frac{1}{1 + y^2} \qquad (2.6.1)$$

This is the Cauchy distribution.

Marsaglia (10, p. 3) discusses the general problem of the properties of the distribution

$$W = \frac{a + X}{b + Y} \qquad (2.6.2)$$

where a and b are non-negative constants and X, Y are independent standard normal variables, N(0, 1). It can be seen that if $W' = X_1/Y_1$ is the ratio of two arbitrary normal random variables, then there are constants c_1 and c_2 such that $c_1 + c_2 W'$ has the same distribution as W. Thus the study of equation (2.6.2) suffices for the general ratio X_1/Y_1.

Starting with the bivariate normal distribution and using the cumulative distribution method, described in Chapter 2.4 of this section, Marsaglia develops an expression for f(t), the p d f of W, or

$$f(t) = \frac{e^{-0.5(a^2+b^2)}}{\pi(1+t^2)} \left[1 + \frac{q}{\varphi(q)} \int_0^q \varphi(y)dy \right] \quad (2.6.3)$$

$$\text{where} \quad q = \frac{b + at}{\sqrt{1+t^2}}, \quad \text{and}$$

φ is the standard normal p d f

This equation must be evaluated numerically, and some results of computer runs are shown in Figure 2.3. This Figure is adapted from (10, p. 5). Note that the quotient of two normal variables is actually bimodal in some cases. Figure 2.4 (10, P. 6) shows that the division between the unimodal and bimodal distribution result depends on where the point (a, b) lies.

Product of $n \leq 10$ Independent Random Normal Variables in Series Form

Springer and Thompson (7, p. 36) derive a result for the product of n normal random variables, for $n \leq 10$. The Mellin transform method is used for solving this problem, and the inversion is accomplished by use of a digital computer. The inversion integral is

$$h(x, \sigma) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \left\{ \frac{2^{\frac{s-1}{2}} \sigma^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \right\}^n ds \quad (2.6.4)$$

A great deal of mathematical effort yields the result

$$h(x, \sigma) = \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left[x^2/(2\sigma^2)^n\right]^{-s}}{(n-1-k)\,|k|\,(2\pi\sigma^2)^{n/2}} \left(\frac{\ln x^2}{(2\sigma^2)^n} \right)^{n-1-k} \frac{d^k}{ds^k} \left[(s+j)^n \Gamma^n(s) \right] \right\}_{s=-j} \quad (2.6.5)$$

where $\ln = \log_e$

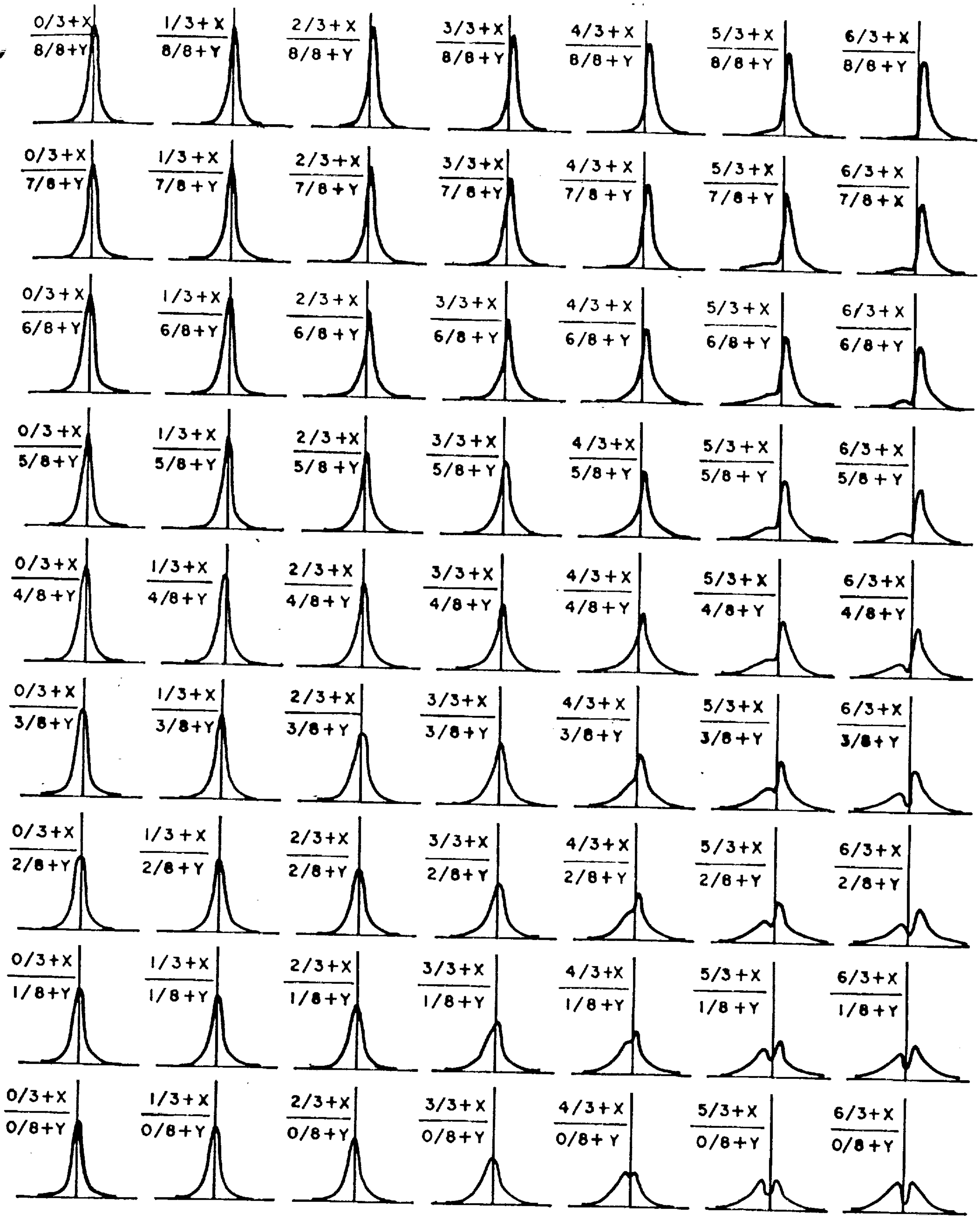

FIGURE 2.3 GRAPHS OF THE DENSITY OF $(a+X)/(b+Y)$, WHERE $a>0$, $b>0$ AND X,Y ARE INDEPENDENT, STANDARD NORMAL VARIABLES. VALUES $a = 0/3, 1/3, \ldots, 6/3$ AND $b = 0/8, 1/8, \ldots, 8/8$ WERE CHOSEN SO AS TO REPRESENT THE POSSIBLE SHAPES OF THE DENSITY FUNCTION.

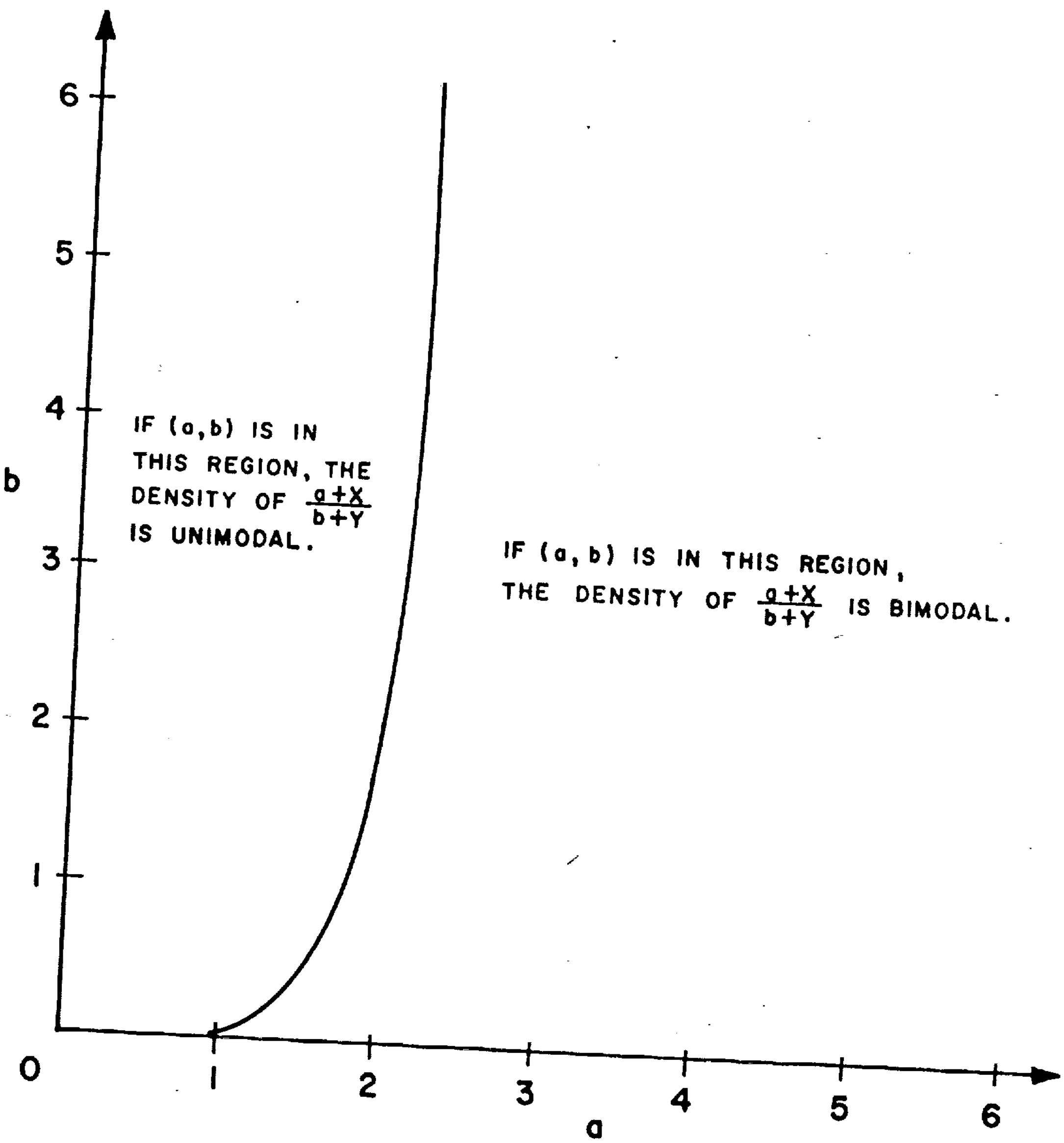

FIGURE 2.4 THE DENSITY OF $(a+X)/(b+Y)$ IS UNIMODAL OR BIMODAL ACCORDING TO THE REGION OF THE POSITIVE QUADRANT IN WHICH THE POINT (a,b) FALLS.

A similar formula (7, p. 47) is given that allows each of the functions to have a different standard deviation σ_i.

Theorems for Functions of the Lognormal Distribution

The lognormal distribution is an interesting one for use in functions of random variables because it is one of the simplest skewed distributions. The lognormal distribution enjoys the same reproductive properties with respect to multiplication and division that the normal distribution enjoys with respect to addition and subtraction. That is to say, the product or quotient of two lognormal distributions is lognormal.

Presented here are theorems regarding the distributions of functions of the lognormal distribution.

1. If X is $L(\mu, \sigma)$ then $1/X$ is $L(-\mu, \sigma)$; (3, p. 10).

2. If X is $L(\mu, \sigma)$ and b and c are constants where $c > 0$, (say $c = e^a$) then cX^b is $L(a + b\mu, b\sigma)$; (3, p. 11).

3. If X_1 and X_2 are independent L variables then the product X_1X_2 is also an L variable;

   if X_1 $\boxed{\text{is}}$ $L(\mu_1, \sigma_1)$
   and X_2 is $L(\mu_2, \sigma_2)$
   then X_1X_2 is $L(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$; (3, p. 11).

4. If $\{X_j\}$ is a sequence of independent L variates where X_j is $L(\mu_j, \sigma_j)$, $\{b_j\}$ a sequence of constants and $c = e^a$ a positive constant, then provided $\sum_j b_j\mu_j$ and $\sum_j b_j^2\sigma_j^2$ both converge, the product $c\prod_j X_j^{b_j}$ is $L(a + \sum_j b_j\mu_j, \sqrt{\sum_j b_j^2\sigma_j^2})$; (3, p. 11).

5. If X_1 is $L(\mu_1, \sigma_1)$ and X_2 is $L(\mu_2, \sigma_2)$, the ratio X_1/X_2 is $L(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$; (3, p. 11).

6. If $X_j (j = 1, \ldots n)$ are independent L variates with the same parameters μ and σ their geometric mean $(\prod_{j=1}^{n} x_j)^{1/n}$ is $L(\mu, \sqrt{\sigma^2/n})$; (3, p. 12).

7. If X_1 and X_2 are two positive independent variates such that X_1X_2 is L variate then both X_1 and X_2 are L variates.

Product and Quotient of Two Independent Random Gamma Variables

Springer and Thompson (7, p. 49) utilize the Mellin transform to obtain the following results.

The gamma random variable is characterized by the p d f ($\eta = 1$)

$$f(x) = \frac{x^{\beta} e^{-x}}{\Gamma(\beta + 1)} \qquad 0 \leq x \leq \infty \tag{2.6.6}$$

$$= 0 \text{ elsewhere}$$

The Mellin transform is

$$M(f(x)) = \frac{\Gamma(s + \beta)}{\Gamma(\beta + 1)} \tag{2.6.7}$$

Thus the Mellin transform of the product $Y = X_1X_2$, where X_1 and X_2 are identically distributed,

$$M(g(y)) = \left(\frac{\Gamma(s + \beta)}{\Gamma(\beta + 1)}\right)^2 \tag{2.6.8}$$

Inverting (2.6.8)

$$g(y) = \frac{2}{\{\Gamma(\beta + 1)\}^2} K_o(2y^{\frac{1}{2}}) \qquad o < y \leq \infty \tag{2.6.9}$$

where $K_o(y)$ denotes Bessel's function of the second kind.

Proceeding in a similar fashion, the p d f $h(y)$ for $Y = X_1/X_2$ is shown to be (7, p. 50), if X_1 and X_2 are identically distributed.

$$g(y) = \frac{\Gamma(2\beta + 2)}{\{\Gamma(\beta + 1)\}^2} y^{\beta} (1 + y)^{-2(\beta + 1)} \tag{2.6.10}$$

Figure 2.5 shows a plot of this function.

Theorems for the Beta Distribution

Some useful properties of the beta distribution have been shown by Jambunathan (9). The beta distribution seems to have potential application in structural reliability theory (Chapter 2.2), so that results for it are of interest:

1. If X is $B_1(a, b)$, and Y is $B_1(a + b, c)$ then XY is $B_1(a, b + c)$; (9, p. 402).
2. If $X_1, X_2, \ldots X_p$ are p independent B_1 random variables with parameters (a_i, b_i) for $i = 1, 2, \ldots, p$, and if $a_{i+1} = a_i + b_i$ for $i = 1, 2, \ldots, (p-1)$ then the product $X_1X_2 \ldots X_p$ is $B_1(a_1, b = \sum_{i=1}^{p} b_i)$; (9, p. 402).
3. If $U = (1 + Y) / (1 + X)$, and if U is B_1 (b-d, d) while Y is B_2 (a,b) then X is $B_2(a + d, b-d)$, provided that U and Y are independent (9, p. 403).

A brief mention will be made of some other works which were not included in the main part of this report because they were not thought to be quite as significant as the ones detailed above.

LeCam (42) discusses the distribution of sums of independent random variables. In addition to their results mentioned above, Springer and Thompson (7) have given results for products, quotients, and geometric means of independent random normal variables, in closed form; products, quotients, and geometric means of independent random Cauchy variables, in series form for general n, and in closed form for $n = 10$; products of $n = 2,3,6$ independent random normal variables in tabular form; quotients of gamma, rectangular, and Cauchy independent random variables; and "mixed" products of $n = 2$, namely, a rectangular - Cauchy, a rectangular - gamma, and a rectangular - normal product.

Marsaglia (10) in addition to his work presented above, gives a solution for ratios of sums of uniform random variables, that is,

$$Y = \frac{U_1 + \ldots + U_n}{V_1 + \ldots + V_m}$$

Donahue (43) discusses general examples of applied problems involving products and quotients of random variables, discusses general theoretical

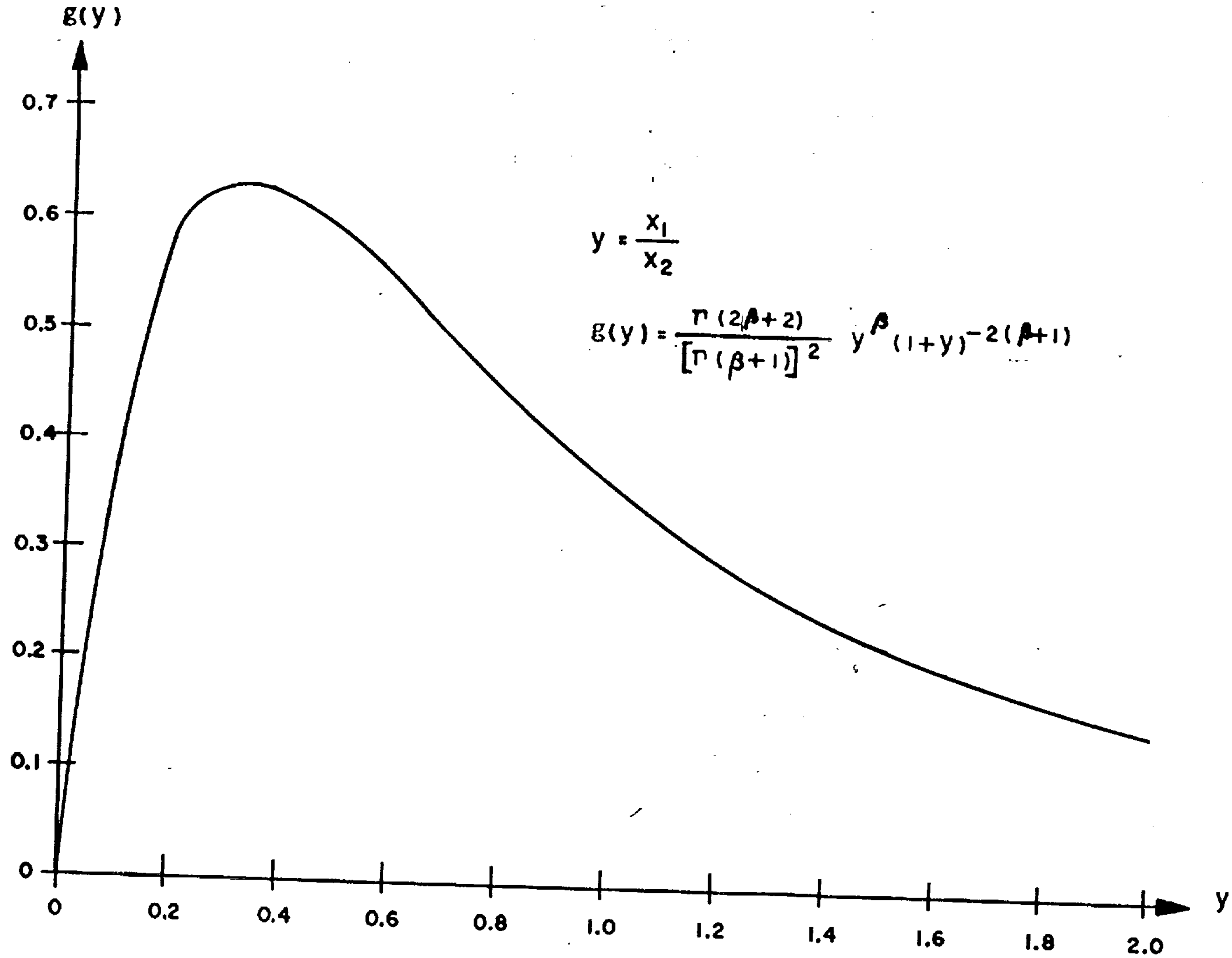

FIGURE 2.5 THE DISTRIBUTION OF THE QUOTIENT OF TWO INDEPENDENT RANDOM GAMMA VARIABLES

models for determining the distribution of products and quotients, gives results pertaining to products and quotients of random variables which generally occur as measurement error, gives limiting distributions and asymptotic results, and has a very extensive annotated bibliography.

DeZur and Donahue (44) discuss integral equation solutions for product and quotient of independent random variables, and also discuss the product of two (not necessarily independent) normal variables. They report limited success in their effort.

Shah (5, p. 2) discusses an application of the lattice distribution (a discrete distribution) in describing the fatigue life of a part.

All of these works contain interesting material which should be explored further.

Other distributions are also mentioned in the literature. The work of Gumbel (21) on the extremal distributions has some mention of fatigue life problems. The double exponential distribution is mentioned by Hayes (6, p. 121) as best describing wind loads on a structure. Freudenthal, Garrelts, and Shinozuka (16, p. 100) and Shinozuka and Nishimura (45) discuss general, or series, representation for distributions. This approach shows promise for numerical techniques.

Summary

Here we have presented some of the more interesting and promising functions of random variables which were uncovered in the literature search. Not all of them are presented; references to other results may be found by consulting Tables 2.1 and 2.2.

Several mathematical methods have been discussed in this section, and all of them probably have some future in the study of functions of random variables for structural application. The Fourier and Mellin transform methods stand out as particularly valuable tools for dealing with sums, differences, products, and quotients, because they can be evaluated by numerical methods when they do not yield closed form solutions.

The Monte Carlo method also stands out as being an important method which will always provide an answer.

Results which apply to structural reliability have been found, and the background has been laid for further work in this area. In particular, the results for the quotient of two normal variables (10), the product of n = 10 normal variables, (7, p. 36), several useful theorems for the lognormal distribution (3), and some work on the product and quotient of gamma variables, (7, p. 49), have been found as well as some theorems for the beta distribution. The above techniques and results give much hope that the problems involved in structural reliability can be solved for distributions that are not normal as well as for those that are normal.

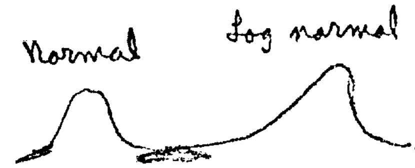

TABLE 2.1

DISTRIBUTION, FUNCTION OF DISTRIBUTION, AND SOURCE REFERENCE

DISTRIBUTION	FUNCTION	REFERENCE (S)
1. Normal	a. Ratio of two independent Normal	(8), (10)
	b. Product of $n \leq 10$ independent Normal	(7)
	c. Sum and difference	(17), (30)
	d. Geometric mean	(7)
	e. General product and quotient	(7), (8), (43), (44)
2. Lognormal	a. $1/X$	(3)
	b. cX^b	(3)
	c. $X_1 \cdot X_2$	(3)
	d. $c \prod_j X_j^{b_j}$	(3)
	e. X_1/X_2	(3)
	f. $(\prod_j X_j)^{1/n}$	(3)
3. Gamma	a. $X_1 \cdot X_2$	(7)
	b. X_1/X_2	(7)
4. Beta	a. $X \cdot Y$	(9)
	b. $X_1 X_2 \ldots X_p$	(9)
5. Mixed		(7)
6. Product and quotients of miscellaneous		(7), (43), (44)
7. General density functions		(16), (45)

TABLE 2.2

MATHEMATICAL METHODS USED IN THE
STUDY OF FUNCTIONS OF RANDOM VARIABLES
AND SOURCE REFERENCES

METHOD	REFERENCES
1. Algebra of Normal Functions	(17), (18), (29)
2. Change of Variables	(11), (12), (13), (25), (30), (37), (43)
3. Moment Generating Function	(3), (11), (12), (13), (30)
4. Fourier Transform	(7), (35), (36), (38), (39)
5. Mellin Transform	(7), (8), (35), (39), (40), (48)
6. Characteristic Function	(3), (41), (43)
7. Cumulative Distribution Function	(10), (43)
8. Monte Carlo	(50), (51), (52)

CHAPTER 2.7

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

As a result of this study, the following conclusions are drawn:

1. Mathematical methods for handling functions of random variables for structural reliability applications do exist.

2. Some results which will be useful for structural reliability applications have been found, and application of the mathematical techniques which have been discussed above should yield others.

3. The Fourier transform, convolution, and inversion method is very promising for sums of independent random variables.

4. The Mellin transform, convolution, and inversion method is very promising for products and quotients of independent random variables.

5. The change of variable method will be limited but may be useful for some functions, provided that the functions and transformations are not too complicated.

6. The moment generating function method and the characteristic function method may be useful in certain cases.

7. The cumulative distribution function method may yield some results for simpler functions, such as $Y = 2X$, etc.

8. The Algebra of Normal Functions Method <u>may</u> provide the best method for engineers to estimate structural reliability without resorting to time-consuming and costly computer studies. The approximations involved in the Algebra of Normal Functions <u>may</u> be close enough to the true distributions involved in most structural reliability problems to form the basis for a good structural reliability estimate. The conjecture must be investigated further.

9. The Monte Carlo Method provides a very flexible and powerful tool for work in this area.

Recommendations

The following recommendations for future efforts are made:

1. Explore the Fourier transform and Mellin transform methods to develop fully their usefulness for functions of random variables for structural reliability applications.

2. Further effort should be made to develop results for functions of distributions such as:

   1. Weibull

   2. Gamma

   3. Beta

   Few results were found for these distributions, and they are of potential importance in the area of structural reliability.

3. Other functions of random variables such as $Y = \ln X$, $Y = e^X$, $Y = \sin X$, etc., should be investigated. These will eventually be necessary for structural reliability work, although the sum, product, quotient, and difference form a useful beginning.

4. Other distributions should be investigated further, namely:

   1. Series Representation

   2. Extremal

   3. Double Exponential

   4. Lattice*

   The applicability of these distributions to structural reliability should be studied, and, if they are thought to be applicable, functions of them should be developed.

5. Further effort should be expended in an attempt to justify the use of various distributions on a physical, or phenomenological basis,

---

*A discrete distribution (5).

rather than on the basis of best fit to data.

6. The question of what is an independent and what is a dependent random variable in structural reliability theory should be examined closely, and care should be taken to apply results for functions of random variables properly, based on whether the variables are independent or dependent.

7. Functions of random variables for mixed (not identically distributed) distributions, such as the product of lognormal and normal distributions, should be studied. Results which have been found for such studies do not seem to be too important to structural reliability (7).

8. The accuracy of the approximations used in The Algebra of Normal Functions, and its value as a rapid and practical method to estimate structural reliability should be assessed and evaluated. (See Ref. 29).

9. The Monte Carlo method should be developed extensively for solving problems in the functions of random variables.

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SECTION 3

DETERMINATION OF FAILURE GOVERNING STRENGTH DISTRIBUTIONS

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CHAPTER 3.1

INTRODUCTION

In the engineering application of the design by reliability methodology, one of the most difficult tasks may be the determination of the actual strength and stress distributions involved. In this Section we wish to consider two methods for determining realistic values for strength distributions. The first method involves the use of functions random variables, as discussed in Section 2. Examples of this method will be the Algebra of Normal Functions and the Monte Carlo technique. The second method will be that of direct experimentation. The appropriate failure governing strength distributions can be generated directly by experimental means. Such an experimental program is now being conducted at The University of Arizona. Finally, a discussion of the use of modifying factors to relate laboratory tests to actual parts in service will be given.

This entire area of determining the actual stress and strength distributions, where such distributions are considered to be the result of applying the methods of functions of random variables of the engineering parameters involved, is an area in which much work remains to be done. Furthermore, there is, indeed, a great need to sell this methodology to practicing engineers, government, and management, so that it will become a standard procedure to think of design as a problem in functions of distributions rather than one of single values.

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CHAPTER 3.2

STRENGTH DISTRIBUTION DETERMINATION BY FUNCTIONS OF RANDOM VARIABLES METHOD

Frequently in engineering practice, it becomes necessary to estimate the fatigue strength of a part going into service. This is currently being done in a conventional manner by methods similar to those discussed in Section 1. Let us now consider the design by reliability approach to this problem using the Algebra of Normal Functions method with the following example:

Example 3.1.-A round rotating member shown in Figure 3.1 is loaded in such a way as to be subjected to a reversed bending moment, M. The fatigue strength of the part is given by

$$S_e = k_a k_b S'_e$$

where:

S_e = endurance limit of the part in service

S'_e = basic endurance limit of the material

k_a = surface finish factor

k_b = size factor

For a design life of greater than 10^6 cycles, the strength distribution can be taken as time invariant. If the design life of the above member is 10^7 cycles, we can calculate the fatigue strength of the members (f(s) in Figure 3.2), using the Algebra of Normal Functions method as follows: Assuming the independent variables in the above equation are normally distributed, their distribution parameters are taken to be

$$\bar{S}'_e = 80{,}000 \text{ psi}, \qquad \sigma_{S'_e} = 6{,}400 \text{ psi}$$

$$\bar{k}_a = 0.70, \quad \sigma_{k_a} = 0.05$$

$$\bar{k}_b = 0.85, \quad \sigma_{k_b} = 0.09$$

The method of Algebra of Normal Functions has been described in Section 2. Using Equations (2.5.3) and (2.5.4) for the product, we have

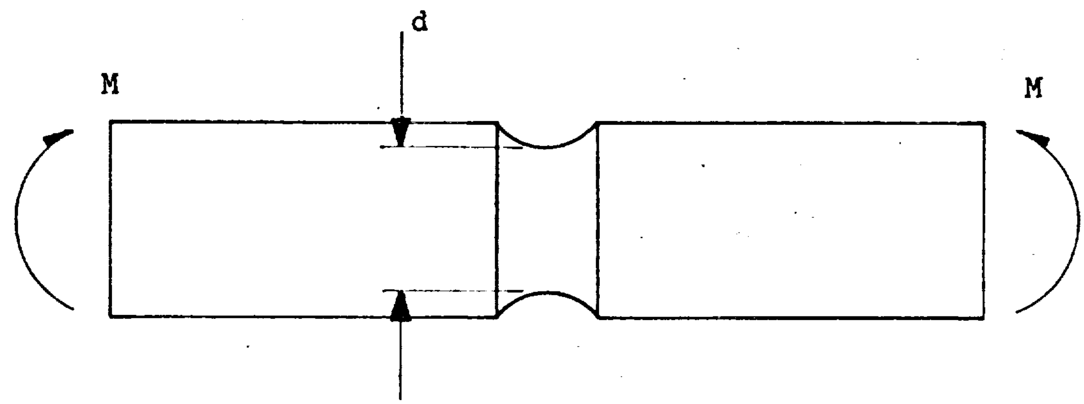


FIGURE 3.1 PART SUBJECTED TO REVERSED BENDING

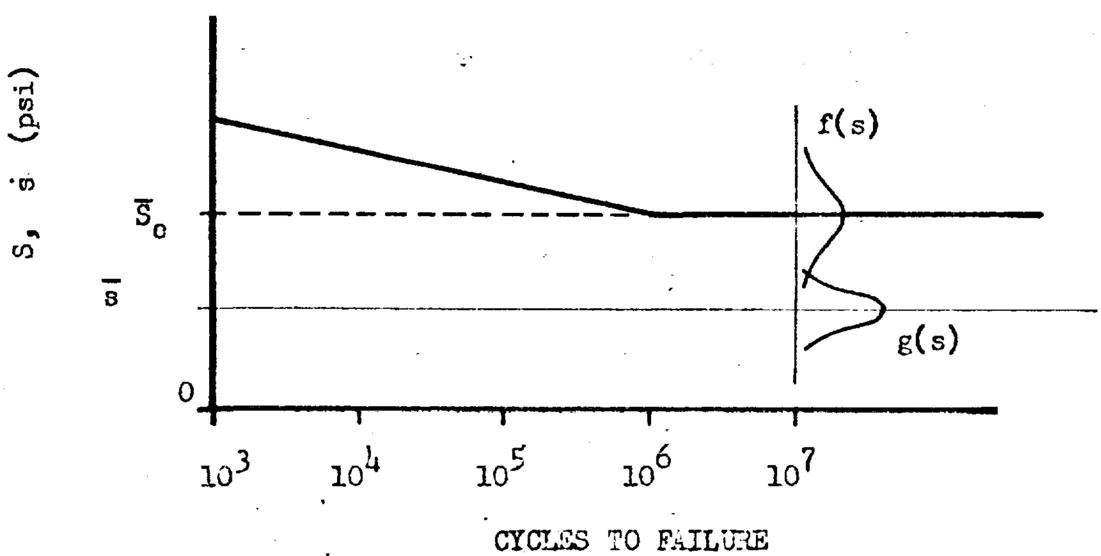


FIGURE 3.2 S-N DIAGRAM DEPICTING THE DESIRED STRENGTH DISTRIBUTION f(S)

$$\overline{k_a k_b} = (\bar{k}_a)\,(\bar{k}_b) = (0.70)\,(0.85)$$

$$\overline{k_a k_b} = 0.595$$

and

$$\sigma_{k_a k_b} = \sqrt{\bar{k}_a^{\,2} \sigma_{k_b}^{\;2} + \bar{k}_b\, \sigma_{k_a}^{\;2} + \sigma_{k_a}^{\;2} \sigma_{k_b}^{\;2}}$$

$$\sigma_{k_a k_b} = \sqrt{(0.70)^2\,(0.09)^2 + (0.85)^2\,(0.05)^2 + (0.05)^2\,(0.09)^2}$$

$$\sigma_{k_a k_b} = \sqrt{0.00580025} \cong \sqrt{0.0058}$$

or

$$\sigma_{k_a k_b} = 0.0761$$

Further

$$\overline{k_a k_b S_e'} = (\overline{k_a k_b})\,(\bar{S}_e') = (0.595)\,(80{,}000)$$

$$\overline{k_a k_b S_e'} = 47{,}600 \text{ psi}$$

and

$$\sigma_{k_a k_b S_e'} = \sqrt{(\overline{k_a k_b})^2\,(\sigma_{S_e'})^2 + (\bar{S}_e')^2\,(\sigma_{k_a k_b})^2 + (\sigma_{S_e'})^2\,(\sigma_{k_a k_b})^2}$$

$$= \sqrt{(0.595)^2\,(6{,}400)^2 + (80{,}000)^2\,(0.0761)^2 + (6{,}400)^2\,(0.0761)^2}$$

$$= (51.687 \times 10^6)^{\frac{1}{2}}$$

thus,

$$\sigma_{k_a k_b S_e'} = 7{,}190 \text{ psi}$$

Therefore, according to the Algebra of Normal Functions, S_e is normally distributed with

$$\bar{S}_e = 47{,}600 \text{ psi}$$

$$\sigma_{S_e} = 7{,}190 \text{ psi}$$

Example 3.2.-Let us repeat Example 3.1, but determine the distribution of the fatigue strength by the Monte Carlo method. Using the program listed in Appendix A of this Report, the following parameters result for the distribution of S_e:

Results of the Monte Carlo Approach
(1000 trials)

Mean, psi	Standard Deviation, psi	Skewness	Kurtosis
47,614	6,843	0.256	3.365

These results compare favorably with those obtained by the Algebra of Normal Functions method. We note that the strength distribution resulting from the Monte Carlo approach is not quite normal, since the exact normal distribution has a skewness of zero and a kurtosis of 3.0. However, the error in approximating the result by a normal distribution is not too serious in this case.

The Algebra of Normal Functions method, within the restrictions listed in Section 2, provides an acceptable method for estimating strength distributions and resulting reliabilities. A more accurate estimate of the parameters of the strength distribution can be obtained by using the Monte Carlo method, but this method demands computer time and the resulting expense.

CHAPTER 3.3

FINDING THE FATIGUE STRENGTH DISTRIBUTION BY DIRECT EXPERIMENT

In some cases, it may be very difficult to come up with good estimates for the means and standard deviations of the fatigue strength and the modifying factors, due to lack of test information and/or engineering experience with the particular application involved. In such a case, it is necessary to conduct a test program on laboratory specimens. The steps required to conduct the tests and analyze the resulting data so that they result in the desired strength distribution will now be explained. Such a test is now being conducted at The University of Arizona. It is believed that such tests and data analysis have not been conducted elsewhere. Therefore, the method will be explained in detail.

Suppose that it is desired to find the strength distribution for a part, such as that shown in Figure 3.3. Details of the loading, material, and desired life are given in this figure. Due to lack of experience with this material in this particular condition, and with this type of loading, it is desired to conduct a test program in order to determine its true strength distribution.

It becomes apparent at the outset that testing machines capable of testing large quantities of 2 in diameter bars are rare, if not non-existent. So the first requirement of the testing program would be to design a test specimen which is smaller than the actual part, but which reproduces as many of the essential features of the actual part as possible. This can be done by scaling the specimen down, but retaining the same stresses, stress concentration factors, stress ratios, and material conditions.

With a fairly large number of such specimens on hand, the testing program would proceed as follows:

First, about 120 specimens are run at various stress levels and an S-N diagram is prepared by generating the life (times-to-failure) distributions at these various alternating stress levels. Notice that for this part

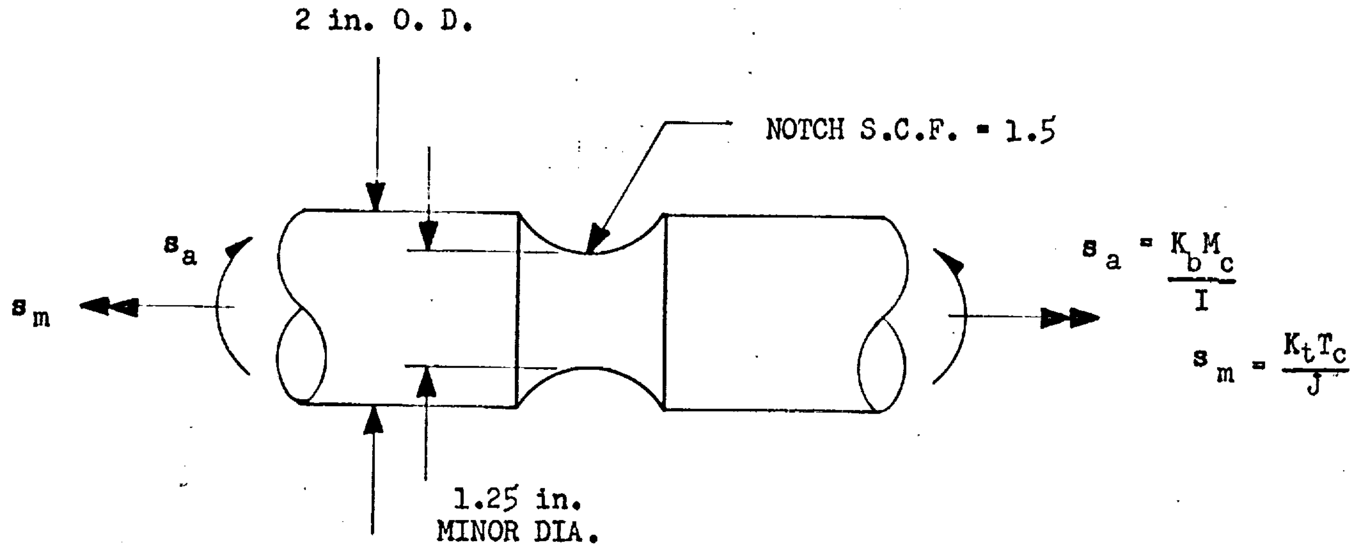
$$s_a = 25{,}000 \text{ psi}$$

and

$$s_m = 12{,}500 \text{ psi}$$

so that

$$\frac{s_a}{s_m} = \frac{25{,}000}{12{,}500} = 2$$



MATERIAL: SAE 4340 STEEL, COND. C-4, COLD DRAWN, ANNEALED BAR STOCK, HEAT TREATED TO ROCKWELL C 35/40

LOADING: s_a = 25,000 psi IN ALTERNATING TENSION AND COMPRESSION;

s_m = 12,500 psi IN SHEAR; ROTATING SHAFT

DESIRED LIFE: 10^5 CYCLES

FIGURE 3.3 DETAILS OF LARGE DIAMETER SHAFT FOR WHICH STRENGTH DISTRIBUTION IS DESIRED

This same stress ratio must be preserved in testing the specimens. The resulting S-N diagram is shown in Figure 3.4.

Once the life distributions have been found for a particular stress ratio, they can be converted to the strength distribution at any desired life by a method which is depicted in Figure 3.5. The cumulative histogram is formed as a cumulative per cent of specimens failing at each stress level. From this cumulative histogram, the strength distribution can be found by conventional statistical methods.

It must be emphasized that this is the fatigue strength for one particular life and one particular stress ratio only. This is emphasized by the plot of the Goodman strength diagram for 10^5 cycles, shown in Figure 3.6. The fatigue strength distribution for $\bar{s}_a/\bar{s}_m = 2$ and 10^5 life cycles is shown in this figure. Also shown are distributions which must be found in order to complete the Goodman fatigue strength diagram. These distributions can be generated by repeating the processes described in Figures 3.4 and 3.5 for different stress ratios, such as

$$\frac{\bar{s}_a}{\bar{s}_m} = 0,\ 1/4,\ 1/2,\ 1,\ 2,\ 4,\ \infty$$

It should also be noted that Goodman strength diagrams can be found for life cycles other than 10^5 by repeating the process described in Figure 3.5 at different life cycles, but for all stress ratios except for stress ratio 0, which corresponds to the static ultimate tensile strength of the specimens.

The University of Arizona's fatigue testing machines, which are described in detail in Section 6 of this Report, are now being used to construct such Goodman fatigue strength surfaces for SAE 4340 steel, treated as described in Figure 1.3.

It is of interest here to estimate the number of specimens required to establish this fundamental property of a material. The minimum number of specimens recommended for each stress level of an S-N diagram is twenty specimens (1, p. 39). Let us say that 18 specimens is an absolute minimum. Referring to Figure 3.5 and Table 7.3, 4 to 6 stress levels need to exist in order to determine the strength distribution at a particular life cycle. Examining Figure 3.6, it seems that to establish an acceptable Goodman fatigue strength diagram, at least 6 distributions at a specified life cycle but for different stress ratios would be required, plus the ultimate tensile strength distribution, therefore, the estimated number of test specimens would be the following :

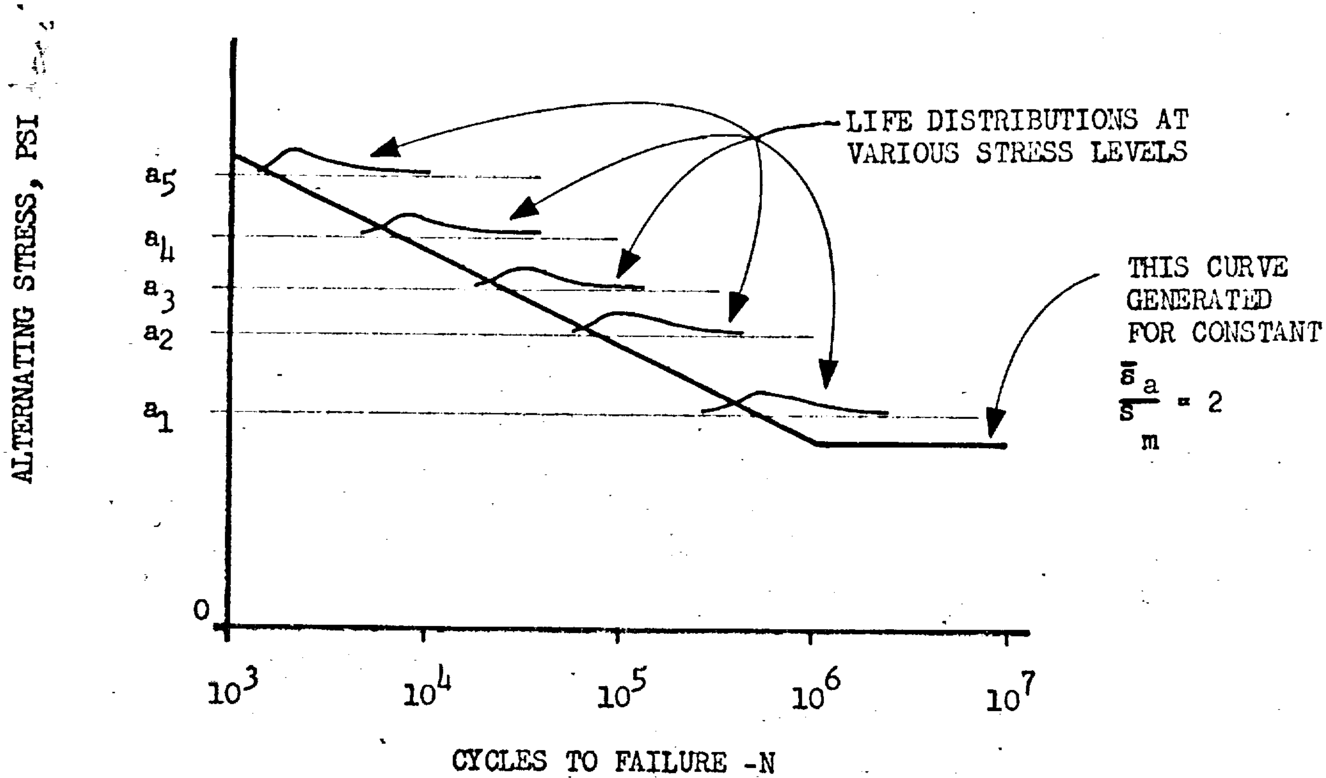


FIGURE ~~3.4~~ 3 DISTRIBUTIONS OF THE TIMES-TO-FAILURE AS FOUND BY FATIGUE TESTS AT CONSTANT STRESS RATIO

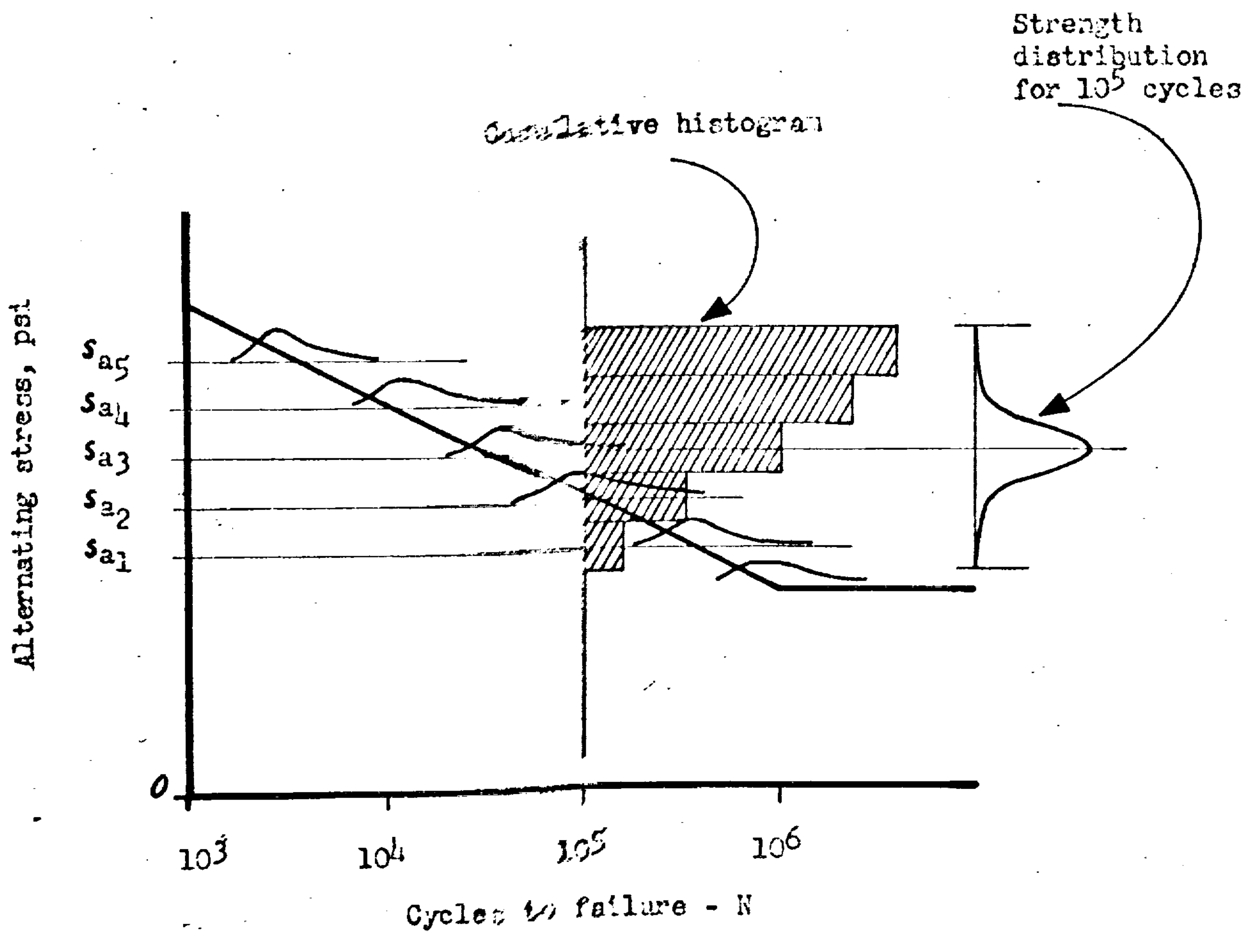


FIGURE ~~3.5~~ 4 THE STRENGTH DISTRIBUTION @ 10^5 CYCLES FOR A STRESS RATIO OF $S_a/S_m = 2$.

For the specific case considered:

$$s_a = \sigma_{xa}$$

$$s_m = \sqrt{3}\,\tau_m$$

Design life: 10^5 cycles

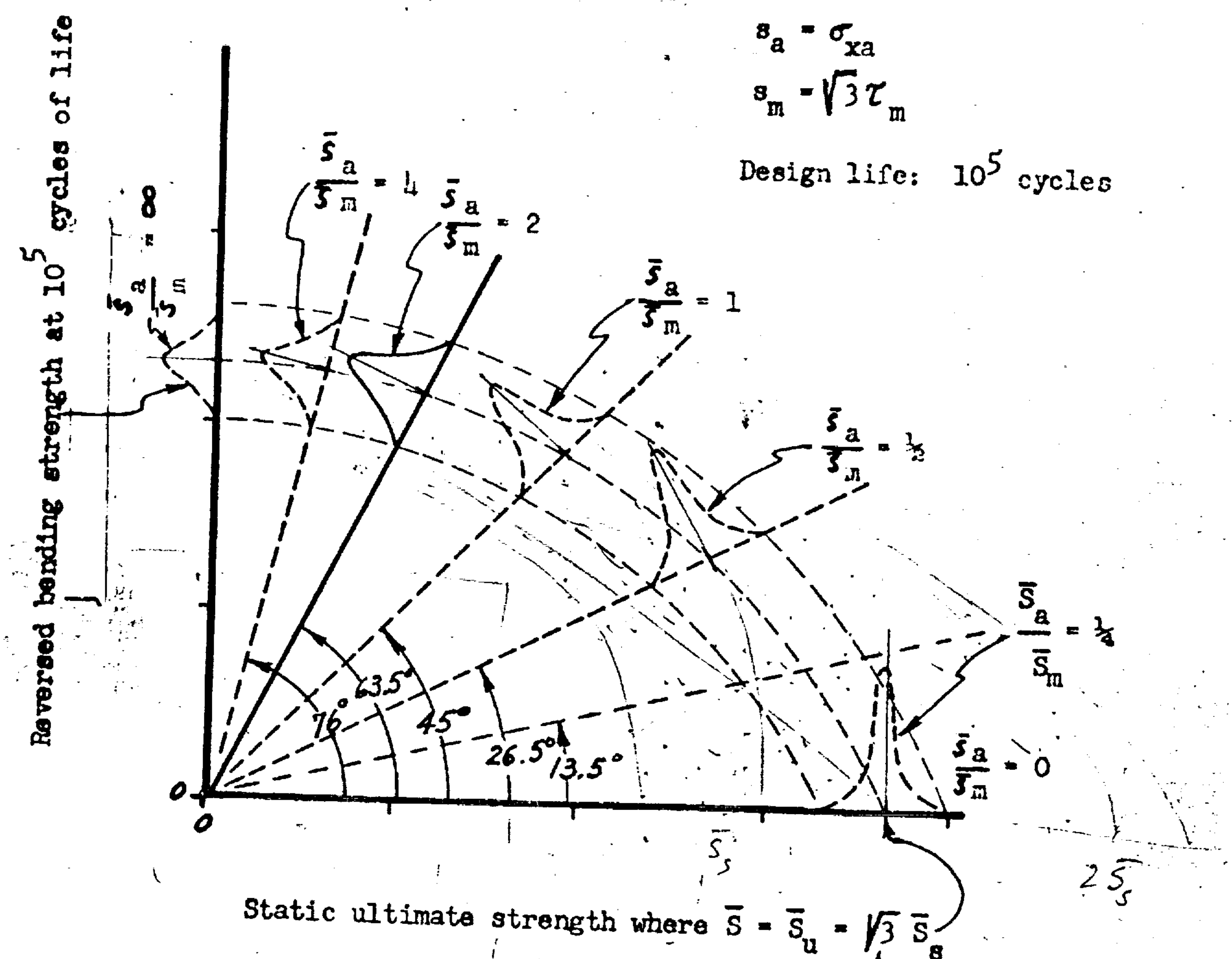


FIGURE ~~3.6~~ 5 GOODMAN FATIGUE STRENGTH DIAGRAM FOR 10^5 CYCLES

$$(\underbrace{5}_{\text{stress levels required}} \times \underbrace{18}_{\text{specimens per stress level}} + \underbrace{30}_{\text{endurance strength run specimens}})\ \underbrace{2}_{\text{stress ratios of } \infty \text{ and } 4} + (4 \times 18 + 30)\ \underbrace{3}_{\text{stress ratios } 2,\ 1, \text{ and } \frac{1}{2}} + (3 \times 18 + 30)\ \underbrace{1}_{\text{stress ratio of } \frac{1}{4}} + \underbrace{18}_{\text{for static ultimate tensile strength distribution}} = 648 \text{ test specimens.}$$

On the other hand, if one is prepared to run a more extensive program in order to determine the distributions more exactly, up to twice as many specimens would be required. In either case, a relatively large number of specimens is the price that must be paid to establish these distributions with good confidence. Nevertheless, the establishment of these distributions is essential to the design by reliability methodology.

The particular problem that we set out to discuss was the determination of the fatigue strength for large diameter shafts. We are now faced with the problem of modifying the results from the test specimens so that they apply to large diameter shafts. This will be discussed next.

CHAPTER 3.4

RELATING RESULTS FROM LABORATORY TESTS TO PARTS IN SERVICE

In the conventional design methodology, the laboratory tested endurance limit is corrected for service conditions by a number of modifying factors (2, p. 166) as follows:

$$S_e = k_a k_b k_c k_d \text{---} k_n S_e' \qquad (3.4.1)$$

This equation can serve as a basis for the design by reliability methodology if it is realized that S_e, S_e' and all of the factors k_a, k_b,...,k_n, must be treated as distributions rather than discrete values. The main problem then becomes that of determining the true distributions of the various factors.

One of the most important considerations for a testing program is to incorporate as many factors as possible into the program itself. Thus, in the example given in Chapter 3.3, the test program was set up so that the factors for surface finish, temperature, notch sensitivity, and heat treatment were eliminated by making their effect on both the test specimen and the part the same. In this example, the only factor necessary to relate the test specimen endurance limit to the part endurance limit is the size factor (2, p. 167).

Now we wish to make some reasonable estimate as to the mean and standard deviation of such a size factor. In the absence of specific test information about this factor, some engineering judgment must be used. Shigley (2, p. 168) gives a figure of 10 to 15 per cent reduction for a 2 in specimen in bending. Let us pick for our 3 in specimen a mean of 15 per cent, a lower value of 10 per cent, and an upper value of 20 per cent. In the absence of specific information to the contrary, it is customary to take such estimated distributions as normal, with 6 σ limits, so that

$$\bar{k}_b = 0.85$$

and

$$6\sigma_{k_b} = 0.20 - 0.10 = 0.10$$

or

$$\sigma_{k_b} = 0.0167$$

Now this distribution of k_b can be combined with the test results to determine the final distribution of S_e for the part. Let us assume for illustrative purposes that as a result of the above-described tests, the fatigue strength of the test specimen could be estimated by

$$\bar{S}'_e = 40{,}000 \text{ psi}$$

$$\sigma_{S_e} = 4{,}000 \text{ psi}$$

Now we will use our estimated distribution of the size factor k_b to correct S'_e to the conditions of the part in service. This can be done by the Algebra of Normal Functions method, where

$$\bar{S}_e = \bar{S}'_e \, \bar{k}_b$$

$$= 40{,}000 \times 0.85$$

$$\bar{S}_e = 34{,}000 \text{ psi}$$

and

$$\sigma_{S_e} \sqrt{\bar{k}_b^{\,2} \, \sigma_{S'_e}^{\,2} + \bar{S}'^{\,2}_e \, \sigma_{k_b}^{\,2} + \sigma_{k_b}^{\,2} \, \sigma_{S'_e}^{\,2}}$$

$$= \sqrt{(0.85)^2 \, (4{,}000)^2 + (40{,}000)^2 \, (0.0167)^2 + (4{,}000)^2 + (0.0167)^2}$$

$$\sigma_{S_e} = 3{,}480 \text{ psi}$$

A Monte Carlo solution to this problem by the program in Appendix A yields the following parameters (for 1,000 trials):

Mean, psi	Standard Deviation, psi	Skewness	Kurtosis
33,887	3,566	0.0497	2.778

Again the Algebra of Normal Functions and the Monte Carlo solution are in good agreement. Also again, the normal approximation involved in the Algebra of Normal Functions shows good agreement with the Monte Carlo solution. That is, the Monte Carlo solution shows a skewness of 0.0497 and a kurtosis of 2.778 which are pretty close to those of the normal distribution of a skewness of zero and a kurtosis of 3.000.

Proceeding in this manner, and also in the manner of Examples 3.1 and 3.2, any number of distributed factors can be incorporated. The problem again lies in determining the true distributions of such factors. This again serves to point up the need for accumulating statistically significant amounts of data regarding these factors. This sort of effort must be reserved for further research.

In the above examples, it might occasionally be possible to use some of the more sophisticated techniques described in Section 2, for example, if the product of lognormal, or the product of n standard normal distributions were involved. It must be stated at the present time, however, that, considering the nature of information available to the designer about the distributions of engineering variables, the Monte Carlo method and the Algebra of Normal Functions method provide the most expedient solutions. The Algebra of Normal Functions method must, of course, be applied within the limitations given in Section 2.

Determining Strength Distributions From Published Data

In a few cases, it may be possible to determine failure governing strength distributions directly from published data by using the method described above. If enough data is available from these tests in terms of cycles-to-failure at various stress levels, and if the data is for test pieces which are similar to the part being designed, then the data can be converted to the desired strength distributions. Unfortunately, it is not often that such data is available.

State-of-the-Art in Present-Day Design and its Relation to Design by Reliability

A representative discussion of present-day design methods and their relationship to the design by reliability methodology is in order at this point. It has been mentioned that the modifying factors which appear in Equation (3.4.1) are usually presented as discrete values, and thus they do not directly supply the information needed for design by reliability. We shall now consider some examples of how these factors are currently found and point out how they may be used for design by reliability.

Size Factor.-In Figure 3.7, Lipson and Juvinall (3, p. 109) present a chart for size-factors, k_b, vs. specimen diameter. Note that for bending and torsion a range is given. In conventional design, one usually uses an "average" k_b of 0.85. This chart can be adapted to structural reliability by making an estimate for the mean and standard deviation of k_b. A normal distribution can be assumed with the mean of k_b taken as the middle of the range, and the range can be assumed to cover 6σ.

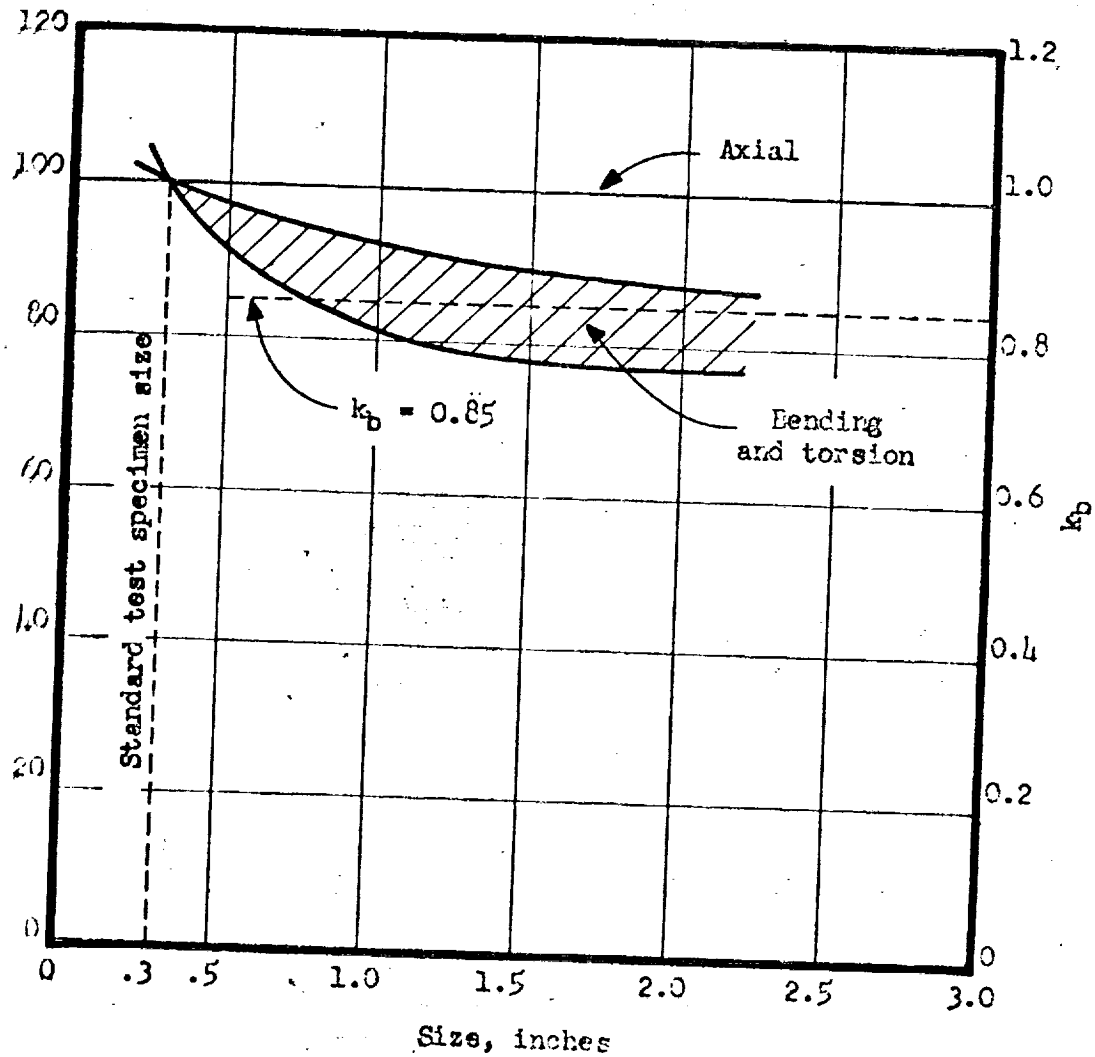


FIGURE 3.7 SIZE EFFECT - BENDING, AXIAL, AND TORSIONAL LOADS (3, p. 109)

Surface Finish Factor.-A graph such as that shown in Figure 3.8 (3, p. 111) is typical of those used to present the surface finish factor. In this case, the surface finish factor (for steels) is found from an appropriate knowledge of the surface condition and tensile strength (on hardness) of the material. In the absence of large numbers of tests on a particular part, this graph is about all the information which the designer has. For purposes of reliability design, this graph represents only a starting point. Faced with this situation, the designer will probably have to assume a normal distribution, take the value from the graph as the mean, and estimate the standard deviation on the basis of experience and judgment.

Shot-Peening Factor.-In Table 3.1 (3, p. 141), the percentage increase in endurance limit for a number of steels is presented. These can easily be converted into shot-peening factors for conventional design. For reliability design, however, these factors are not too useful, and, again, about the best the designer can do is assume a normal distribution, use the given value as the mean, and estimate the standard deviation.

Cold Rolling and Cold Stretching Factor.-Table 3.2 (3, p. 142) gives the percentage increase in the endurance limit of steels due to cold rolling and cold stretching. These can also be converted to "factors" for conventional design. The situation for reliability design is the same incomplete one as mentioned before under shot-peening factor.

Quenching and Flame-Hardening Factor.-Figure 3.9 (3, p. 149) shows the effects of quenching and flame-hardening on the endurance limit of a particular steel. Such a figure may be used to compute modifying factors for conventional design. The situation for reliability design is the same incomplete one as mentioned before.

Corrosion Factor.-Table 3.3 (3, p. 152) permits the computation of conventional correction factors for a typical steel. From a reliability standpoint, the approach would be the same as previously mentioned.

Plating Factor.-Table 3.4 (3, p. 152) permits the computation of conventional correction factors. Again, for use in design by reliability, the above-mentioned estimates and assumptions would have to be made.

The above examples are representative of the information available for the conventional design approach. A large number of references, for example (4 through 16), are available for estimating these factors for almost any design situation. However, from the design-by-reliability standpoint, these factors leave a great deal to be desired. Much work now remains to be done in order to present these factors, not as single values, but as distributions.

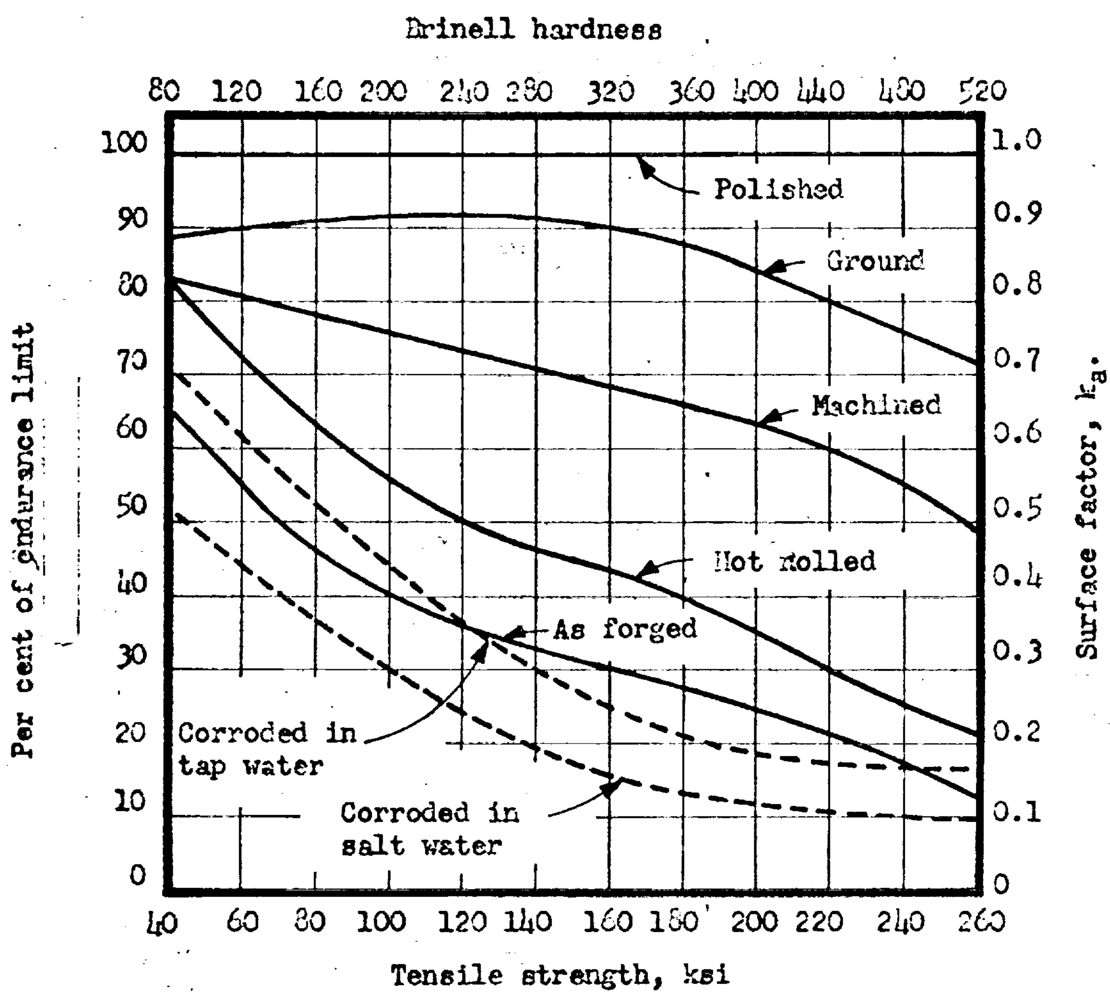


FIGURE 3.8 REDUCTION OF ENDURANCE STRENGTH DUE TO SURFACE FINISH (3, p.111)

TABLE 3.1

EFFECT OF SHOT PEENING ON ENDURANCE LIMIT - SPECIFIC TEST VALUES (3, P. 141)

Material	Specimen	Treatment Prior To Shot Peening	Surface Prior to Shot Peening	Percentage Increase in Endurance Limit
SAE 1020 Steel	Plate	As-rolled	Polished	9
SAE 1045 Steel	Plate	Normalized	Polished	11
SAE 1050 Steel	Plate	As-rolled	Polished	22
SAE S4340 Steel	Standard	Quenched-and-drawn	Polished	18
Ni-Cr-Mo Steel	Standard	Carburized	Polished	4
Alloy Steel	Bar	Hardened	Polished	2
Alloy Steel	Bar	Hardened	Machined	23
SAE 1020 Steel	Plate	As-rolled	Hot-rolled	34
SAE 1045 Steel	Plate	Induction hardened	Not-rolled	50
Rail Steel	Rail	As-rolled	Hot-rolled	32
0.65 C Steel	Wire	As-drawn	Hot-rolled	42
SAE 1095 Steel	Wire	As-drawn	Hot-rolled	50
Ni-Cr-Mo Steel	Standard	Carburized	Hot-rolled	23
NE 9470 Steel	Standard	Carburized	Hot-rolled	50
NE 9240 Steel	Standard	Carburized	Hot-rolled	53
NE 8650 Steel	Axle	Quenched-and-tempered	As-forged	100
NE 8650 Steel	Axle	Normalized-and-tempered	As-forged	54
NE 8650 Steel	Flat bar	Quenched-and-tempered	Severely ground	90
4340 Steel	Shaft	Quenched-and-tempered	Chrome plating*	90
Phosphor bronze	Coil spring			40
Beryllium copper	Coil spring			80
S-816(Co-Cr-Ni Base)	Coil spring			80
18-8 Stainless	Coil spring			70
13-2 Stainless	Coil spring			50

* Shot peening performed before chrome plating.

TABLE 3.2

EFFECT OF COLD ROLLING AND COLD STRETCHING ON ENDURANCE LIMIT - SPECIFIC TEST VALUES (3, P. 142)

Material	Specimen	Treatment Prior to Cold Working	Surface Prior to Cold Working	Percentage Increase in Endurance Limit
		Cold rolling		
SAE 1045	Bar	Normalized	Polished	6
SAE 1045	Bored	Quenched-and-tempered	Polished	52
SAE 1045	Bored	Quenched-and-tempered	Polished	33
SAE 1045	Bar	Normalized	Machined	27
SAE 1045		Notched	Machined	120
SAE 1045		Notched	Machined	52
SAE 1050	Press fit	Normalized	Machined	150
3.1 Ni	Press fit	Normalized	Machined	100
0.35 C	Thread	Quenched-and-tempered	Machined	33
0.20 C	Bars	Hot-rolled	Hot-rolled	67
Alloy steel	Shaft with fillet	Normalized-and-tempered	Polished	68
Alloy steel	Shaft with fillet	Normalized-and-tempered	Polished	56
Alloy steel	Shaft with fillet	Quenched-and-tempered	Polished	30

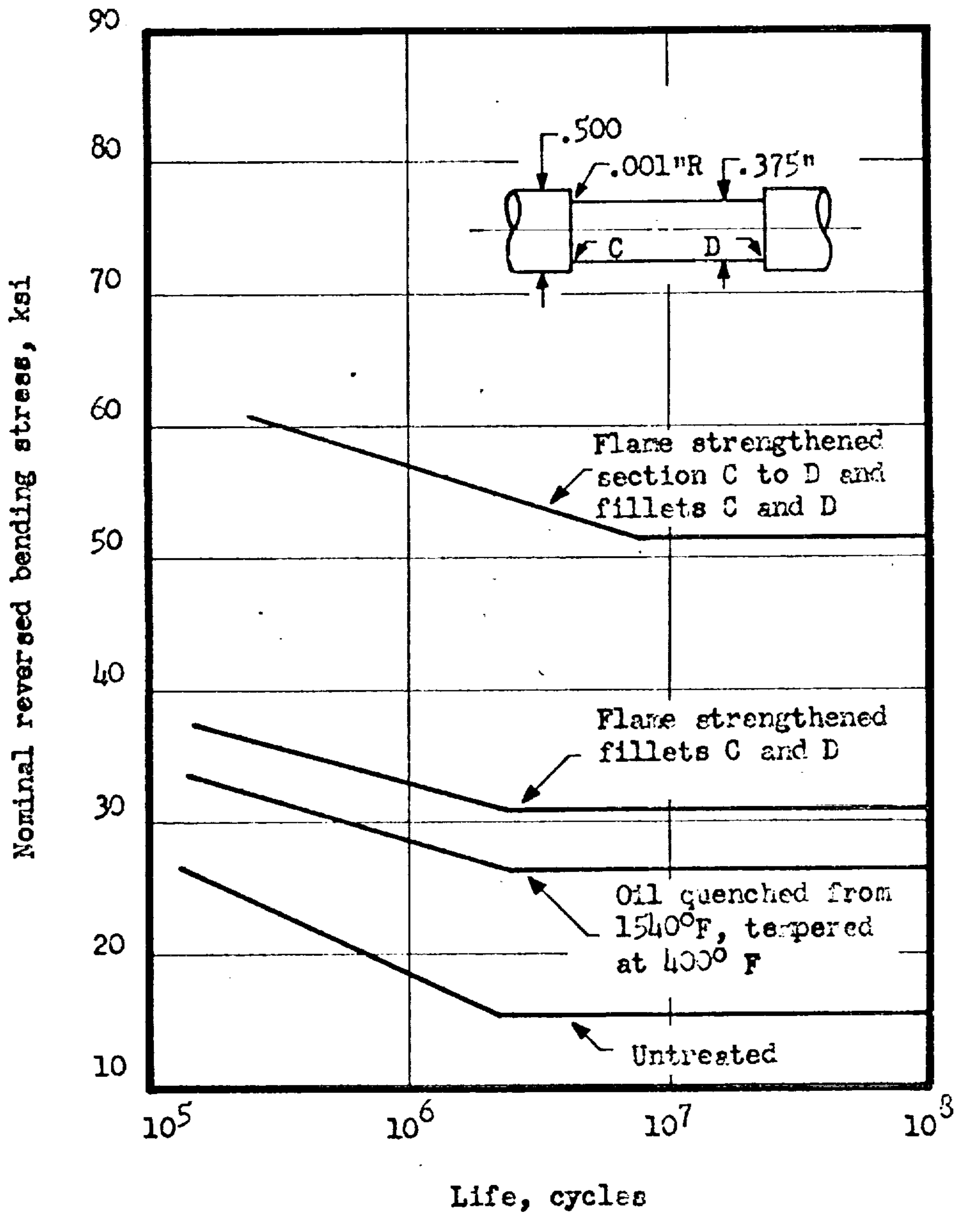


FIGURE 3.9 EFFECT OF FLAME STRENGTHENING ON NOTCHED SAE 1045 STEEL BARS (3, p. 149)

TABLE 3.3

EFFECT OF FRESH WATER CORROSION ON ENDURANCE LIMIT OF A TYPICAL STEEL (3, P. 152)

Treatment	Endurance Limit in Air psi	Endurance Limit in Fresh Water psi	Percentage Decrease due to Corrosion
Uncoated	31,000	15,500	50
Copper plated	28,000	28,000	0
Nickel plated	23,500	23,500	0
Chromium plated	33,000	33,000	0

TABLE 3.4

FATIGUE STRENGTH OF CHROMIUM PLATED PARTS (3, P. 152)

Steel	Treatment	Plating Thickness in.	Endurance Limit psi	Endurance Limit Percentage Decrease due to Plating
Cr-Mo-V		None	74,000	0
Cr-Mo-V	Plated 15 hr.	0.0015	68,000	8
Cr-Mo-V	Plated 8 hr.	0.006	64,000	14
Cr-Mo-V	Plated 8 hr., tempered 250°C	0.008	31,000	58
Cr-Mo-V	Plated 1 hr., tempered 250°C	0.0015	62,000	16
SAE 6130	Normalized, not plated	None	33,000	0
SAE 6130	Normalized, plated	0.00018	30,000	9
SAE 6130	Normalized, plated	0.0045	32,000	3
SAE 6130	Quenched-and-drawn, not plated	None	65,500	0
SAE 6130	Quenched-and,drawn, plated	0.00015	38,000	57
SAE 6130	Quenched-and-drawn, plated	0.0045	41,000	38

As a preliminary step in this direction, we should point to the work of Haugen (17) who has reported strength properties of metal alloys in statistical form, giving the means and standard deviations of reported data. Studies have to be conducted to determine the true nature of the underlying statistical distributions to this data and then to determine the true parameters of these distributions.

DISCUSSION

In the design problems worked out in this Section, the strength distributions have been taken as normal. Although this is a practice used in structural reliability, it is not strictly accurate in many cases. The assumption of the normal distribution is customarily made for two reasons:

(1) Lack of sufficient data for making a better decision as to which distribution to use.

(2) The ease of working mathematically with the normal distribution.

It should be pointed out that there is really no basis, in many cases, to justify this assumption. One of the areas in structural reliability which needs thorough research is that of determining the true strength distributions and their parameters.

SUMMARY

This Section has provided the engineer with the basic tools, both analytical and experimental, for determining the failure-governing strength distributions of parts in combined-stress fatigue. In the next Section, we will provide a parallel discussion for the case of determining the distribution of the failure-governing stress.

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SECTION 4

DETERMINATION OF FAILURE GOVERNING STRESS DISTRIBUTIONS

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CHAPTER 4.1

INTRODUCTION

The problem of determining the failure governing stress is analagous to the problem of determining the failure-governing strength. However, instead of determining material properties and factors for modifying these properties, we are now concerned with determining distributions of loads, dimensions, load factors, stress concentration factors, etc., for the part in service. Again, the realistic determination of these distributions may be difficult, since most of them are commonly reported by their mean value only, or, at best, as a range of minimum to maximum values.

Once their distributions are determined, their synthesis into the failure governing stress is accomplished by the same techniques as those described in Section 3 for synthesizing the failure governing strength distribution.

In this section we will indicate how the engineer can determine in practice the distributions for the various loads, dimensions and factors and then synthesize them into the failure governing stress. An example will also be given.

CHAPTER 4.2

LOAD DISTRIBUTIONS

As was mentioned in Section 2, even so simple a failure governing stress distribution as

$$\bar{s} = \frac{\bar{P}}{\bar{A}}$$

requires that the load distribution, P, and the area distribution, A, be known in order to treat s as a distribution. Therefore, in any structures problem, one must first determine values for the distributions of the loads acting on the part. Estimates of the loading distributions may come from many sources, among them are:

1. Instrumenting actual parts in service.

2. Instrumenting actual parts in the laboratory.

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3. Instrumenting and testing test specimens, which simulate the part, in the laboratory.

4. Obtaining field service data on existing similar equipment and adjusting the results to those to be expected on the actual parts.

5. Using engineering judgment and experience.

Again, it must be stated that to find today actual test data in sufficient quantity and accuracy to make a confident decision about the "correct" distributions of loads involved (and their parameters) would be the exception rather than the rule. Therefore, in many cases it will be necessary to rely on engineering judgment and experience in order to make a realistic estimate of the load distributions.

CHAPTER 4.3

DIMENSION DISTRIBUTIONS

Usually it is not too difficult to make realistic estimates for the distributions of dimensions. In the absence of quality control information and data, it is customary to assume that the distribution of a dimension, such as the diameter of a shaft, is normal, and that the tolerance of the distribution spans 6 standard deviations. Thus, the dimension 0.520 in to 0.500 in would be assumed to have a normal distribution with a mean of 0.510 in, and with a standard deviation of $\sigma = \frac{(0.520 - 0.500)}{6} = 0.0033$ in.

Sometimes a much better estimate can be made, based on data from the Quality Control Department. Records may be available on a large number of similar parts which have been manufactured by similar processes.

It is, of course, impossible to obtain actual dimensions from the manufactured part while the part is at the design level, but it is usually possible to make a good estimate of the distributions of the important dimensions of the part.

CHAPTER 4.4

LOAD FACTOR DISTRIBUTIONS

The distributions of load factors will probably be difficult to obtain, since usually these are reported as mean values only. If enough test data is available for statistical analysis, this method can be used. If a range for the factors is given, then a 6σ limit can be assumed to cover this range. In many cases, engineering experience and judgment will have to be used to estimate these distributions.

By definition, the load factor, k_1 is given by

$$k_1 = \frac{P_a}{P_s}$$

where

$$P_a = \text{actual load}$$

and

$$P_s = \text{static load}$$

Here the distributions of P_a and P_s will have to be determined, or $f(P_a)$ and $f(P_s)$ respectively. Subsequently the distribution of k_1, $f(k_1)$ can be determined using the techniques discussed in Section 2, for the distribution of the quotient of two random variables P_a and P_s.

CHAPTER 4.5

DISTRIBUTIONS OF STRESS CONCENTRATION FACTORS

Ordinarily, stress concentration factors are reported as mean values only. However, since the stress concentration factor can usually be represented by a function of the geometry of the part, it may be possible to make a good estimate of the distribution of the stress concentration factor by treating it as a statistical function of the part's dimensions. Naturally, the resulting estimate will be no better than the estimate of the dimensions, but, as was previously mentioned, good estimates for the distributions of these dimensions can be obtained relatively easily.

CHAPTER 4.6

ILLUSTRATIVE EXAMPLES

An example problem follows which illustrates some of the above techniques.

Example 4.1.-The stress in a member subjected to a bending load and having a stress raiser, Figure 4.1, can be calculated from

$$s = K_f \frac{Mc}{I}$$

where

$$K_f = 1 + q\,(K_t - 1)$$

and it is the actual stress concentration factor.

In this example the stress distribution will be calculated using the Algebra of Normal Functions method. Therefore, the variables will be assumed to be normally distributed, with the following parameters:

$$\bar{M} = 2\,860 \text{ in-lb}, \quad \sigma_M = 280 \text{ in-lb}$$

$$\bar{K}_t = 1.5, \quad \sigma_{K_t} = 0$$

$$\bar{q} = 0.80, \quad \sigma_q = 0$$

$$\bar{d} = 1.00 \text{ in.}, \quad \sigma_d = 0.01 \text{ in.}$$

In the equation for the stress in the member in question

$$c = d/2$$

$$I = \pi d^4/64$$

Therefore

$$s = K_f \frac{Md(64)}{(2)\,(\pi)\,d^4} = K_f \frac{32(M)}{\pi d^3}$$

Starting with the equation for K_f we have

$$\overline{qK_t} = (\bar{q})\ (\bar{K}_t) = (0.80)\ (1.5) = 1.2$$

and $$\sigma_{qK_t} = \sqrt{\bar{q}^2\sigma_{K_t}^{\ 2} + \bar{K}_t^{\ 2}\sigma_q^{\ 2} + \sigma_q^{\ 2}\sigma_{K_t}^{\ 2}} = \sqrt{(0.80)^2(0)^2 + (1.5)^2(0)^2 + (0)^2(0)^2} = 0$$

Also
$$(\overline{qK_t - q}) = \overline{qK_t} - \bar{q} = 1.2 - 0.8 = 0.4$$

and
$$\sigma_{(qK_t-q)} = \sqrt{\sigma_{qK_t}^{\ 2} + \sigma_q^{\ 2}} = 0 + 0 = 0$$

Then
$$\bar{K}_f = 1 + (\overline{qK_t - q}) = 1 + 0.4 = 1.4$$

which gives the final distribution

$$\bar{K}_f = 1.4 \text{ and } \sigma_{K_f} = 0$$

Now to solve for the distribution of $\left(\frac{M}{d^3}\right)$

we use the power formula for d^3 (1, p. 4)

$$\bar{d}^3 = (\bar{d})^3 = (1.00)^3 = 1.00$$

and
$$\sigma_{d^n} = n\bar{d}^{n-1}\sigma_d$$

or
$$\sigma_{d^3} = (3)(\bar{d}^2)(0.01) = (3)(1)^3(0.01) = 0.03$$

Then
$$(M/d^3) = \bar{M}/\bar{d}^3 = 2\ 860/1 = 2\ 860$$

and
$$\sigma_{(M/d^3)} = \frac{1}{\bar{d}^3}\sqrt{\frac{(\bar{M})^2(\sigma_{d^3})^2 + (\bar{d}^3)^2(\sigma_M)^2}{(\bar{d}^3)^2 + (\sigma_{d^3})^2}}$$

$$\sigma_{(M/d^3)} = \frac{1}{1}\sqrt{\frac{(2\ 860)^2(0.03)^2 + (1)^2(280)^2}{(1)^3 + (0.03)^2}}$$

$$\sigma_{(M/d^3)} = \sqrt{\frac{8\ 586 \times 10^4}{1.0009}} = (8\ 586 \times 10^4)^{\frac{1}{2}}$$

or

$$\sigma_{M/d^3} = 293$$

and

$$(\overline{K_f M/d^3}) = (\bar{K}_f)(\overline{M/d^3}) = (1.4)(2\ 860)$$

$$\overline{K_f M/d^3} = 4\ 010$$

Then,

$$\bar{s} = \frac{32}{\pi}\ (K_f M/d^3) = \frac{32}{\pi}\ (4\ 010) = 40\ 800 \text{ psi}$$

$$\bar{s} = 40\ 800 \text{ psi}$$

and

$$\sigma_{(K_f M/d^3)} = \sqrt{(\bar{K}_f)^2 (\sigma_{M/d^3})^2 + (\overline{M/d^3})^2 (\sigma_{K_f})^2 + (\sigma_{M/d^3})^2 (\sigma_{K_f})^2}$$

$$\sigma_{(K_f M/d^3)} = \sqrt{(1.4)^2 (293)^2 + (2\ 860)^2 (0)^2 + (293)^2 (0)^2}$$

$$= (169\ 400)^{\frac{1}{2}} = 411 \text{ psi}$$

But

$$\sigma_s = \frac{32}{\pi}\ \sigma_{(K_f M/d^3)}$$

Therefore

$$\sigma_s = \frac{32}{\pi}\ (411) = 4\ 190 \text{ psi}$$

$$\sigma_s = 4\ 190 \text{ psi}$$

Thus,

s is normally distributed with

$\underline{\bar{s} = 40\ 800 \text{ psi}}$

$\underline{\sigma_s = 4\ 190 \text{ psi}}$

Example 4.2.-For purposes of comparison, a Monte Carlo solution to the same problem gives the following parameters (for 2 000 trials):

Mean, psi	Standard Deviation, psi	Skewness	Kurtosis
40 426	4 197	0.042	2.91

This illustrates the good agreement between the Monte Carlo and the Algebra of Normal Functions method when very small variabilities are chosen for σ_M and σ_d.

In determining the failure governing stress distributions for other cases, methods similar to those given previously can be used. In some cases, the use of the more exact results for products, quotients, etc., of distributions, as given in Section 2, might be required. From the design by reliability standpoint, the Algebra of Normal Functions and the Monte Carlo methods present the most expeditious approaches.

CHAPTER 4.7

STATE-OF-THE-ART IN PRESENT-DAY DESIGN AND ITS RELATION TO DESIGN BY RELIABILITY

Present-day design methods for failure governing stress factors are very similar to those for failure governing strength factors. Such factors are usually presented as discrete values and do not directly supply the information needed for design by reliability. Some examples of these current methods are given next.

Stress Concentration Factor.-Many references exist which give stress concentration factors for various configurations under various loadings (2-5). Lipson and Juvinall (6, p.222) present many charts for stress concentration factors. An example is given in Figure 4.1. Here the stress concentration factor is presented as a single value, for use in conventional design. For purposes of design by reliability, the value obtained from the chart can be assumed to be the mean of a normal distribution, and an estimate has to be made for the standard deviation. Or an attempt can be made to compute the stress concentration factor as a function of random variables as described earlier in this Section.

Notch Sensitivity Factor.-In the equation

$$K_f = 1 + q(K_t - 1)$$

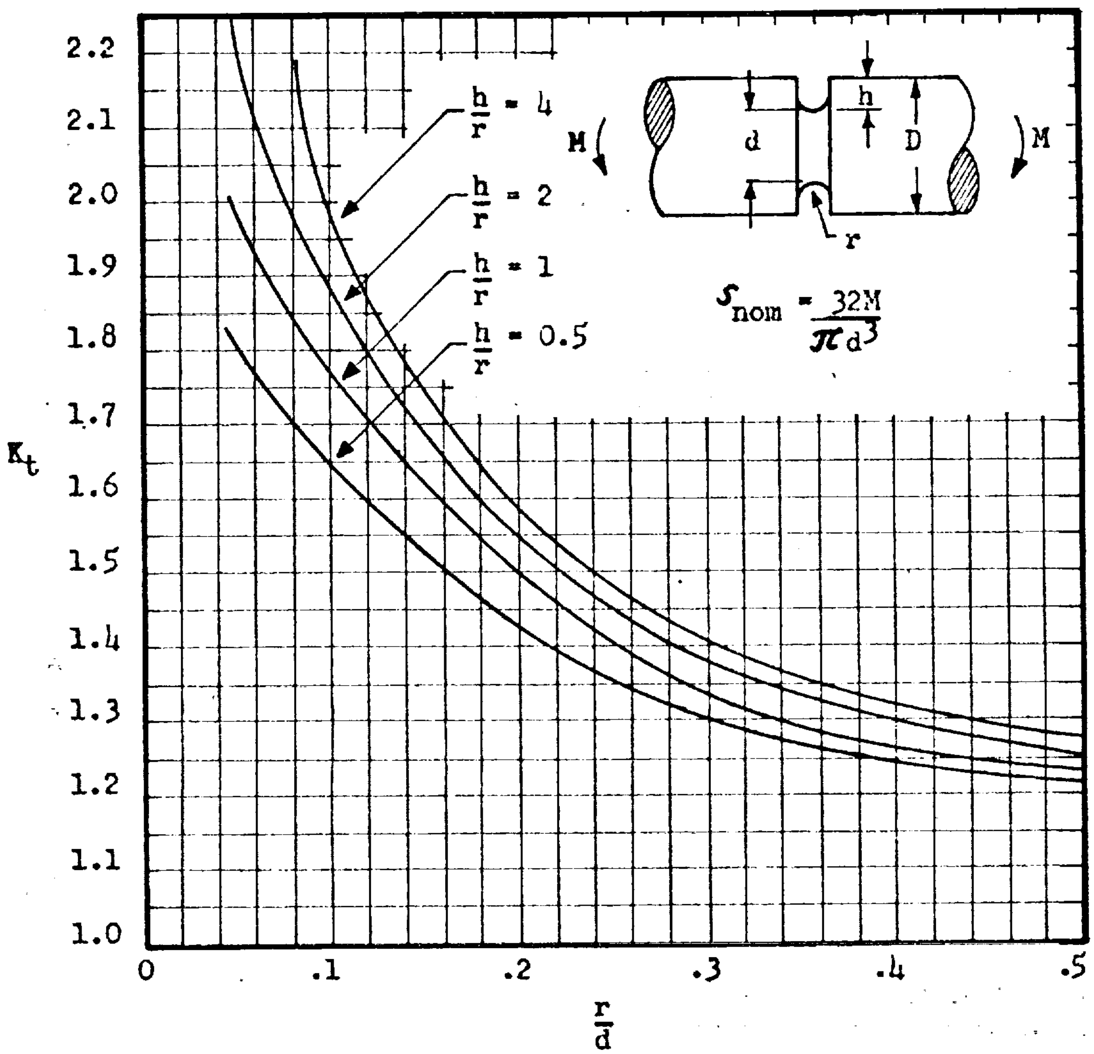

FIGURE 4.1 SOLID CIRCULAR SHAFT CIRCULAR GROOVE BENDING STRESS CONCENTRATION FACTORS (6, p. 222)

the notch sensitivity factor, q, is used to relate the theoretical stress concentration factor K_t, to the fatigue stress concentration factor K_f. Lipson and Juvinall (6, p. 117) present a chart, Figure 4.2, which is representative of those which can be used to find q for conventional design. The situation as far as design by reliability is concerned is the same as that described above for stress concentration factors.

A large number of references do exist (7-29) for stress factors for the conventional design approach. As in the case for failure governing strength factors, the failure governing stress factors given therein are not directly suitable for design by reliability. There is much room for research in this area in order to put these factors on a statistical basis.

DISCUSSION

In Example 4.1, we assumed that the failure governing stress distribution is normal. The same objections apply to this practice as to the use of the normal distribution for the failure governing strength distribution in Section 3. Again, much data needs to be generated to determine accurately the true stress distribution.

SUMMARY

Methods have been presented which will permit the engineer to determine, or estimate, the distributions of variables and factors involved and then arrive at the failure governing stress distribution.

Two of the most useful methods from the standpoint of design by reliability are found to be the Algebra of Normal Functions and the Monte Carlo methods.

Once the distributions for both the failure governing strength and the failure governing stress are determined, the only remaining problem is to determine the resulting reliability. This will be discussed in the next section.

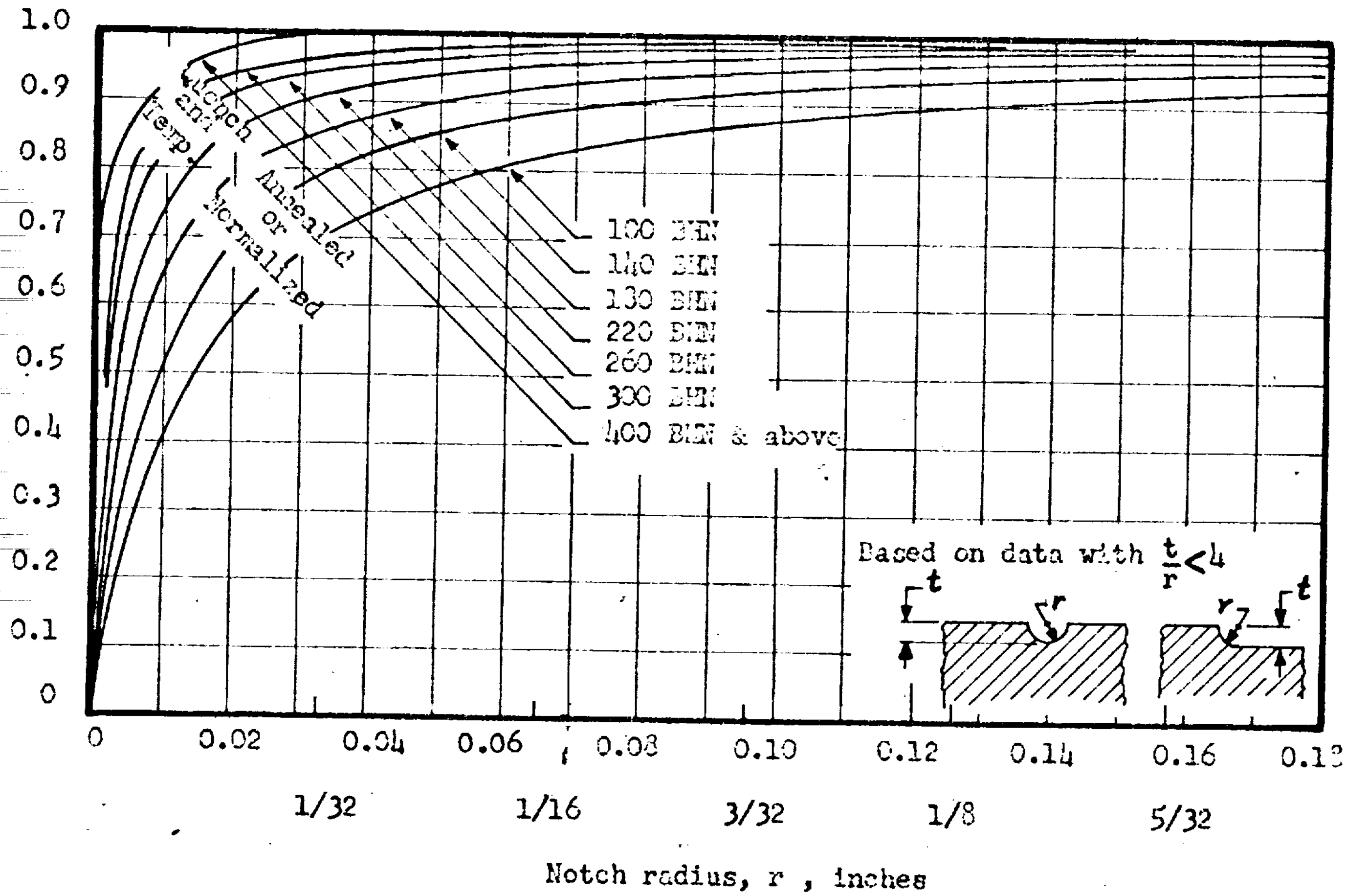

FIGURE 4.2 NOTCH SENSITIVITY FACTORS (6, p. 117)

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SECTION 5

BRIDGING THE GAP
BY RELIABILITY THEORY

CHAPTER 5.1

INTRODUCTION

Finding the reliability when the stress and strength distributions are normal has been discussed in Section 1. We can illustrate this again by considering the stress and strength distributions found in Sections 3 and 4, examples 3.1, 3.2, 4.1, and 4.2.

Using the Algebra of Normal Functions solution, we have, from equations (1.2.19 b and c)

$$t = \frac{\bar{s} - \bar{S}_e}{\left[\sigma_{S_e}^{\;2} + \sigma_s^{\;2}\right]^{1/2}} = \frac{(40{,}800) - (47{,}600)}{\left[(7{,}190)^2 + (4{,}190)^2\right]^{1/2}}$$

$$t = \frac{-\,6{,}800}{(51.7 \times 10^6 + 17.6 \times 10^6)^{\frac{1}{2}}}$$

$$t = \frac{-6{,}800}{8{,}320} = 0.816$$

from which

$$R = 0.793$$

For a comparison, we can use the Monte Carlo solutions

$$\bar{S} = 47{,}614$$

$$\sigma_S = 6{,}843$$

$$\bar{s} = 40{,}926$$

$$\sigma_s = 4{,}197$$

$$t = -\frac{47{,}614 - 40{,}926}{\left[(6{,}843)^2 + (4{,}197)^2\right]^{1/2}}$$

$$t = -\,0.838$$

from which

$$R = 0.799$$

As would be expected, the agreement between the Algebra of Normal Functions solution and the Monte Carlo solution is quite good in this particular case.

When the stress and strength distributions are not normal, a number of techniques are available for solution. These will be discussed next.

CHAPTER 5.2

DETERMINATION OF RELIABILITY WHEN LOGNORMAL STRESS AND STRENGTH DISTRIBUTIONS ARE INVOLVED

A relatively common distribution in reliability studies is the log-normal distribution. It is used when the stress or the strength is markedly skewed in its distribution and when a plot of the natural logarithm of s or S versus their frequency of occurrence is normal.

In other words, if

$$\log_e s = s' \qquad (5.2.1)$$

then for s' to be normally distributed, its density function should be

$$f(s') = \frac{1}{\sigma_{s'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{s' - \bar{s}'}{\sigma_{s'}}\right)^2} \qquad (5.2.2)$$

where

$$\bar{s}' = \frac{\sum_{i=1}^{N} \log_e s_i}{N} \qquad (5.2.3)$$

$$\sigma_{s'} = \sqrt{\frac{\sum_{i=1}^{N} (\log_e s_i)^2 - N(\bar{s}')^2}{N - 1}} \qquad (5.2.4)$$

and N = number of observations.

The stress density function would then be (1, p. 89)

$$f(s) = \frac{1}{s\,\sigma_{s'}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\log_e s - \bar{s}'}{\sigma_{s'}}\right]^2} \quad \text{for } s \geq 0$$

Similar expressions can be written for strength, S.

To determine the reliability when both stress and strength distributions are log-normal we may use the fact that the logs of stress and strength are normally distributed, as discussed before. Then the method given in Section 1 of this report may be used because

$$h(\zeta') = f(S' - s') \qquad (5.2.1)$$

where

$h(\zeta')$ = difference distribution of two normal variates S' and s', which is normal itself.

$$S' = \log_e S \qquad (5.2.2)$$

$$s' = \log_e s \qquad (5.2.3)$$

Consequently

$$\bar{\zeta}' = \bar{S}' - \bar{s}' \qquad (5.2.4)$$

and

$$\sigma_{\zeta'} = \sqrt{\sigma_{S'}^{\,2} + \sigma_{s'}^{\,2}} \qquad (5.2.5)$$

then

$$h(\zeta') = \frac{1}{\sigma_{\zeta'}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\zeta' - \bar{\zeta}'}{\sigma_{\zeta'}}\right)^2} \qquad (5.2.6)$$

The reliability, with lognormal stress and strength distributions, is therefore given by

$$R = \int_{0}^{\infty} h(\zeta') \, d\zeta' \qquad (5.2.7)$$

To evaluate this integral the transformation relating ζ' and the standardized variable t may be used, which is

$$t = \frac{\zeta' - \bar{\zeta}'}{\zeta'} \qquad (5.2.8)$$

The new limits of the integral are for

$$\zeta' = 0 \quad , \quad t = -\frac{\bar{\zeta}'}{\zeta'}$$

and for

$$\zeta' = \infty \quad , \quad t = \infty$$

Therefore

$$R = \int_{-\frac{\bar{\zeta}}{\sigma_\zeta}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt \qquad (5.2.9)$$

Now the value of R can be obtained from available tables of areas under the standardized normal density function, as illustrated in Section 1.

If f(s) is normally distributed and f(S) is lognormally distributed, or vice versa, the methods given in the next three chapters should be used to determine the reliability.

CHAPTER 5.3

DETERMINATION OF RELIABILITY FOR GENERALLY DISTRIBUTED STRESSES AND STRENGTHS

The reliability of a component can be determined from the basic concept that a no-failure probability exists when a given strength value is not exceeded by stress. The probability that a stress of value s_1 exists in interval ds is equal to the area of the element ds or to A_1 on Fig. 5.1 or

$$P\left(s_1 - \frac{ds}{2} \leq s \leq s_1 + \frac{ds}{2}\right) = f(s_1)\ ds = A_1 \qquad (5.3.1)$$

The probability of strength exceeding s_1 is equal to the shaded area A_2, or

$$P(S > s_1) = \int_{s_1}^{\infty} f(S)\ dS = A_2 \qquad (5.3.2)$$

The probability of no failure, i.e., the reliability, at s_1 is the product of these two probabilities, or

$$dR = f(s_1)\ ds \times \int_{s_1}^{\infty} f(S)\ dS \qquad (5.3.3)$$

The component reliability would then be all probabilities of strength being greater than all possible values of stress, (2), (3) or

$$R = \int dR = \int_{-\infty}^{\infty} f(s) \left[\int_{s}^{\infty} f(S)\ dS\right] ds \qquad (5.3.4)$$

The reliability can also be written as (3)

$$R = \int_{-\infty}^{\infty} f(S) \left[\int_{-\infty}^{S} f(s)\ ds\right] dS \qquad (5.3.5)$$

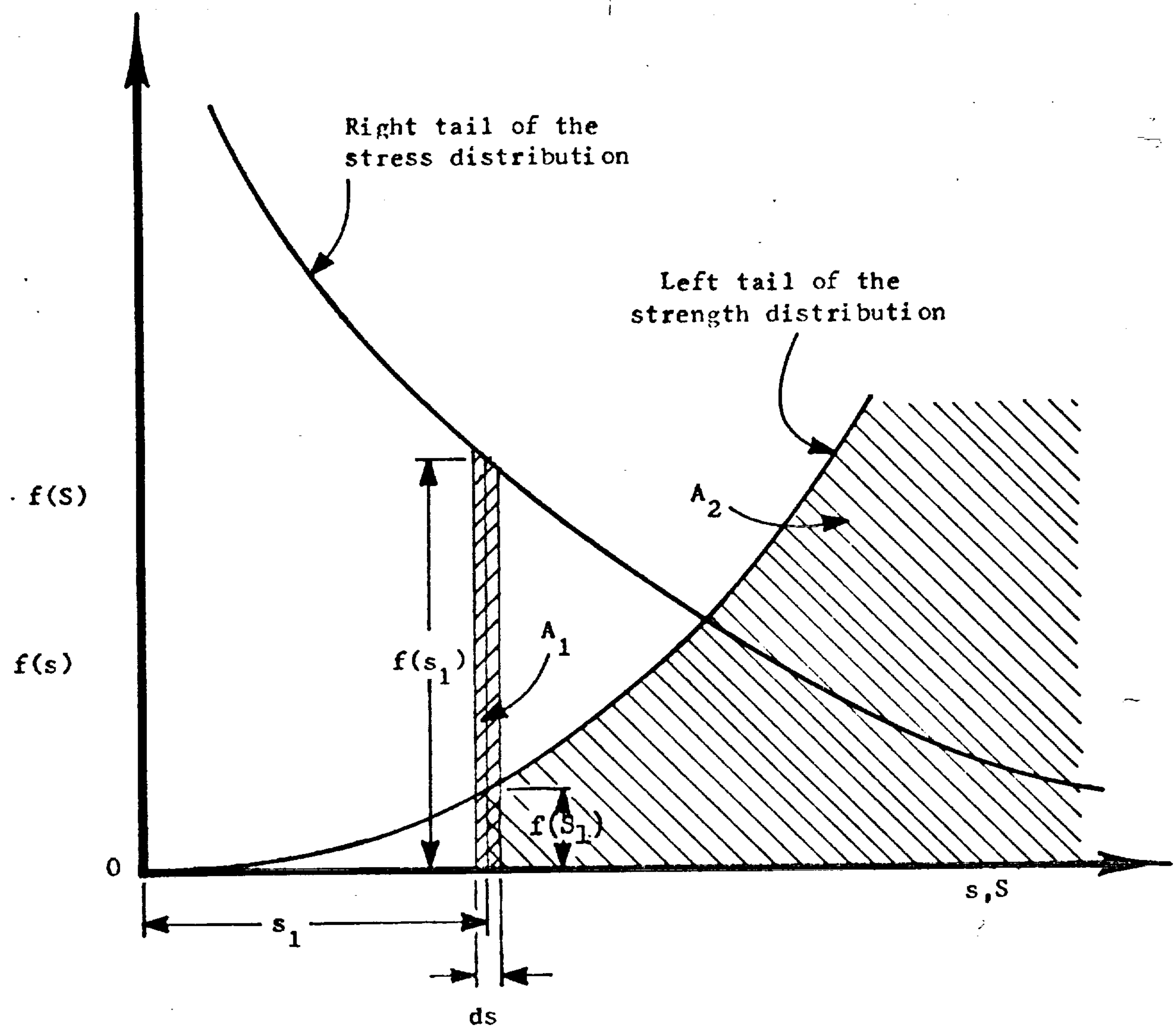


FIGURE 5.1 PROBABILITY AREAS OF STRESS AND STRENGTH FOR RELIABILITY DETERMINATION WHEN STRENGTH EXCEEDS STRESS.

Furthermore, similar expressions can be written for the unreliability, Q, of a component as follows (3):

$$Q = \int_{-\infty}^{\infty} f(s) \left[\int_{-\infty}^{s} f(S)\, dS \right] ds \qquad (5.3.6)$$

or

$$Q = \int_{-\infty}^{\infty} f(S) \left[\int_{S}^{\infty} f(s)\, ds \right] dS \qquad (5.3.7)$$

Equations (5.3.4) and (5.3.5) can now be used to calculate the reliability of any component whose f(s) and f(S) are known. If these distributions are normal or lognormal these equations, by transformation of variables, revert to those discussed in the previous chapter. It must be mentioned that these equations carry limits of integration applicable to distributions which exist from $-\infty$ to $+\infty$. Otherwise these limits should be replaced by the lowest and highest values the random variables can assume. If they are not, then the Transform Method or a computer program utilizing Simpson's Rule may be employed to evaluate these equations and thereby determine reliability or unreliability.

CHAPTER 5.4

TRANSFORM METHOD OF DETERMINING RELIABILITY WITH NON-NORMAL STRESS AND STRENGTH DISTRIBUTIONS

A technique has been developed (3) which consists of a Transform Method for solving the general reliability equation.

The reliability expression

$$R = \int_{-\infty}^{\infty} \left[\int_{s}^{\infty} f(S)\, dS \right] f(s)\, ds \qquad (5.3.4)$$

can be rewritten by letting

$$G(S) = \int_{s}^{\infty} f(S)\, dS \qquad (5.4.1)$$

and

$$F(s) = \int_{s}^{\infty} f(s)\, ds \qquad (5.4.2)$$

Then

$$R = \int_{0}^{1} G\, dF \qquad (5.4.3)$$

where

$$dF = f(s)\, ds \qquad (5.4.4)$$

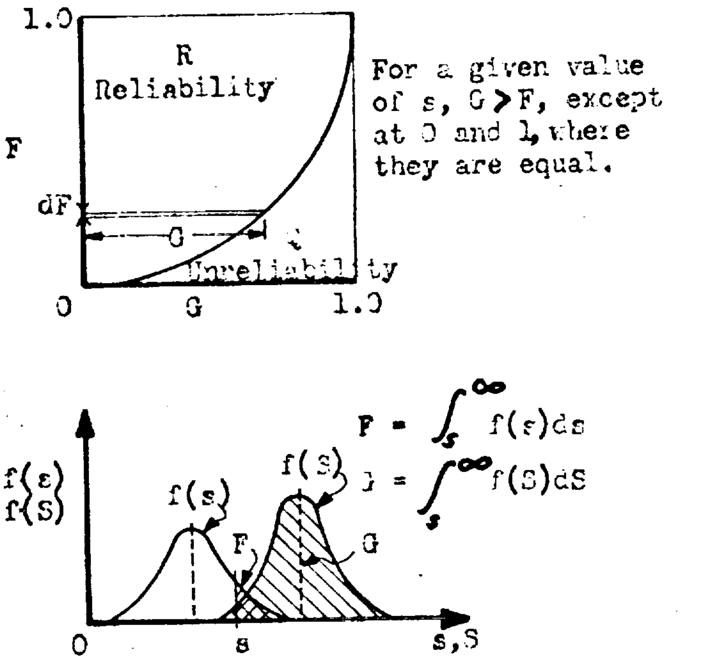
The limits have changed because the variables of integration have changed. Where the lower and upper values of stress are $-\infty$ and ∞ and of S are s and ∞, respectively, the range for either G or F is from 0 to 1 by definition of these new variables. Figure 5.2a shows a plot of F vs. G, the new variables. Inspection of Equation (5.4.3) reveals that the reliability is the area under the $G = f(F)$ curve, and is designated in the upper graph in Fig. 5.2a. This area may be planimetered, and its ratio to the total area bound by the axes, and $F = 1$ and $G = 1$ calculated, thus obtaining the reliability.

The unreliability in this case is obtained from

$$Q = \int_{-\infty}^{\infty} \left[\int_{S}^{\infty} f(s)\, ds \right] f(S)\, dS \qquad (5.3.7)$$

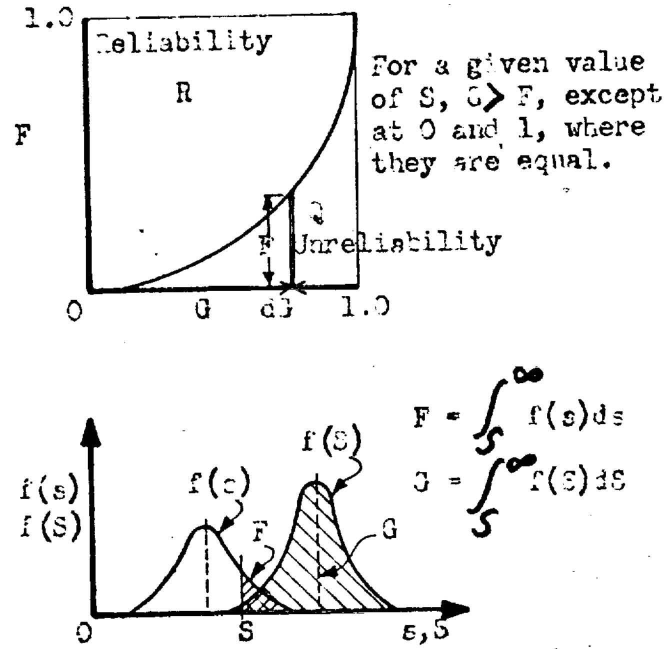
by letting

$$G(S) = \int_{S}^{\infty} f(S)\, dS \qquad (5.4.5)$$



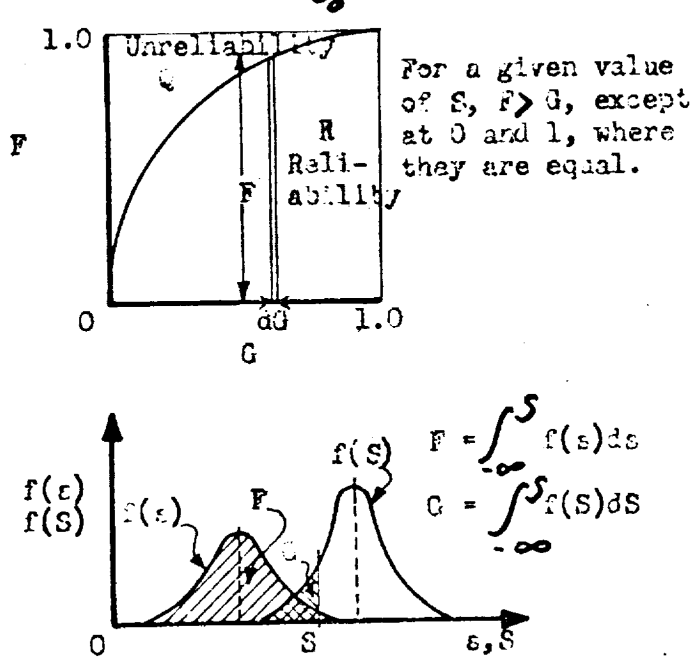
(a) Transform method of reliability determination for any stress and strength distribution where

$$R = \int_0^1 G\,dF$$



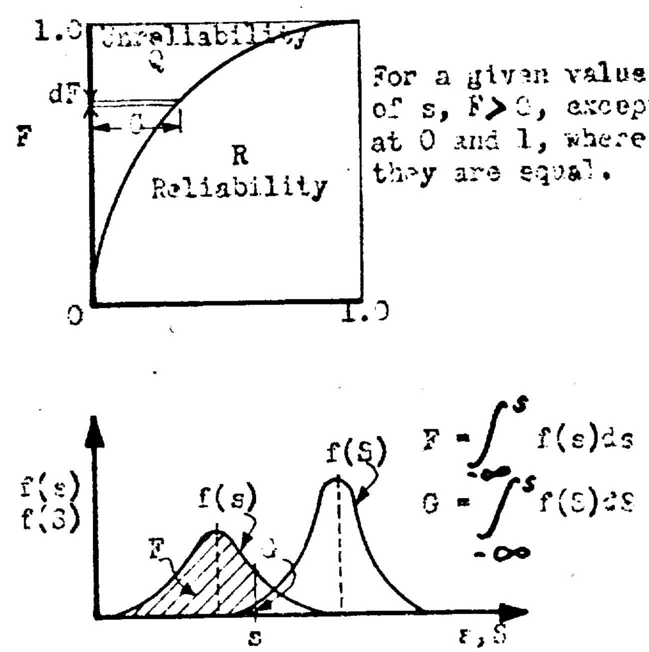
(b) Same as figure 5.2 (a) except for

$$Q = \int_0^1 F\,dG$$



(c) Same as figure 5.2 (a) except for

$$R = \int_0^1 F\,dG$$



(d) Same as figure 5.2 (a) except for

$$Q = \int_0^1 G\,dF$$

FIGURE 5.2 TRANSFORM METHOD OF RELIABILITY DETERMINATION

and

$$F(s) = \int_{S}^{\infty} f(s)\, ds \tag{5.4.6}$$

Then

$$Q = \int_{0}^{1} F dG \tag{5.4.7}$$

where

$$dG = f(S)\, dS \tag{5.4.8}$$

Figure 5.2b shows the plot of F vs. G and the areas giving unreliability and reliability. An alternative expression can be written for the reliability using

$$R = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{S} f(s)\, ds \right] f(S)\, dS \tag{5.3.5}$$

and letting

$$F(s) = \int_{-\infty}^{S} f(s)\, ds \tag{5.4.9}$$

and

$$G(S) = \int_{-\infty}^{S} f(S)\, dS \tag{5.4.10}$$

so that

$$R = \int_0^1 F dG \qquad (5.4.11)$$

where

$$dG = f(S)\, dS \qquad (5.4.12)$$

This reliability is illustrated in Figure 5.2c. Finally a second expression may be written for the unreliability which is

$$Q = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{s} f(S)\, dS \right] f(s)\, ds \qquad (5.3.6)$$

By setting

$$G(S) = \int_{-\infty}^{s} f(S)\, dS \qquad (5.4.13)$$

and

$$F(s) = \int_{-\infty}^{s} f(s)\, ds \qquad (5.4.14)$$

Equation (5.3.6) becomes

$$Q = \int_0^1 G dF \qquad (5.4.15)$$

where

$$dF = f(s)\, ds \qquad (5.4.16)$$

This unreliability is illustrated in Fig. 5.2d.

This method enables the evaluation of reliability or unreliability for any distribution of stress or strength and for any combination of two different distributions of stress and strength, provided the partial areas, F and G, under these distributions can be found.

The methods of finding areas under the relativoly common distributions of gamma and Weibull will be presented later. This way the reliability of components having stress and strength distributions that are normal, lognormal, gamma or Weibull may be determined.

The accuracy of determining the reliability by this transform method depends on the accuracy of evaluating the partial areas F and G, of plotting them and measuring the area for R or Q. The desired accuracy may be obtained by evaluating the areas F, G, and R or Q by digital computers.

Another digital computer method for evaluating R and Q for any distribution, using Simpson's Rule, is presented later.

Determination of Areas Under the Gamma Distribution for Reliability Calculations

Another distribution which arises in reliability is the gamma distribution whose density function is given by

$$f(s) = \frac{e^{-s/\eta}\, s^{\beta}}{\Gamma(\beta+1)\ \eta^{\beta+1}} \qquad (2.3.13)$$

where Γ stands for the gamma function and should not be confused with the density function of the distribution itself.

Partial areas may be found as follows:

$$P(0 < s < s_1) = \frac{1}{\Gamma(\beta+1)\ \eta^{\beta+1}} \int_0^{s_1} e^{-s/\eta}\, s^{\beta}\, ds \qquad (5.4.17)$$

Let $s = \eta v$, then $ds = \eta\, dv$, and when $s = 0$, $v = 0$ and when $s = s_1$, $v = s_1/\eta$. Substitution of these in Equation (5.4.17) gives

$$P(0 < s < s_1) = \int_0^{s_1/\eta} \frac{e^{-v} v^{\beta} dv}{\Gamma(\beta + 1)} \qquad (5.4.18)$$

Pearson (4) tabulates for $p = \beta$

$$I(u,\beta) = \int_0^{s_1/\eta} \frac{e^{-v} v^{\beta} dv}{\Gamma(\beta + 1)} \qquad (5.4.18a)$$

where

$$u = y/\sqrt{\beta + 1} \qquad (5.4.19)$$

Hence

$$P(0 < s < s_1) = I(u,\beta) \qquad (5.4.19a)$$

with

$$s_1/\eta = y = u\sqrt{\beta + 1} \qquad (5.4.20)$$

therefore

$$u = \frac{s_1}{\eta\sqrt{\beta + 1}} \qquad (5.4.20a)$$

Knowing β, η and s_1, calculating u, and entering Pearson's tables with these values of u and β gives the numerical value of Equation (5.4.17).

The following procedure may be used to determine β and η from stress distribution data:

$$\beta = \frac{4}{b} - 1 \qquad (5.4.21)$$

where

$$b = \frac{m_3^2}{m_2^3} \qquad (5.4.21a)$$

$$m_2 = \frac{\sum_{i=1}^{N} (s_i - \bar{s})^2}{N} \tag{5.4.21b}$$

$$m_3 = \frac{\sum_{i=1}^{N} (s_i - \bar{s})^3}{N} \tag{5.4.21c}$$

and

$$\eta = \frac{\sigma_s}{\sqrt{\beta + 1}} = \frac{m_2^{1/2}}{\sqrt{\beta + 1}} \tag{5.4.22}$$

Combining Equations (5.4.20a) and (5.4.22) gives

$$u = \frac{s_\ell}{\sigma_s} \tag{5.4.23}$$

As an example, for a given gamma stress distribution let $\sigma_s = 3{,}000$ psi and $m_3 = 21.7 \times 10^9$. It is required to find the area under this distribution from $s = 0$ to $s_\ell = 8{,}000$ psi. Then

$$u = \frac{s_\ell}{\sigma_s} = \frac{8{,}000}{3{,}000} = 2.67$$

and

$$m_2 = \sigma_s^{\,2} = (3{,}000)^2 = 9 \times 10^6$$

$$b = \frac{(m_3)^2}{(m_2)^3} = \frac{(21.7 \times 10^9)^2}{(9 \times 10^6)^3} = 0.645$$

Therefore

$$\beta = \frac{4}{b} - 1 = \frac{4}{0.645} - 1 = 5.2$$

The tables for u = 2.67 and β = 5.2 give $I(u,\beta) = 0.621 = \int_0^{s_1} f(s)\,ds$ by interpolation.

The gamma stress distribution parameters, η and β, for the example may be found as follows:

$$\beta = 5.2$$

and

$$\eta = \frac{\sigma_s}{\sqrt{\beta + 1}} = \frac{3,000}{\sqrt{5.2 + 1}} = 1210$$

Therefore

$$f(s) = \frac{e^{-s/1210}\, s^{5.2}}{\Gamma(6.2)\, 1210^{6.2}} = \frac{1}{2.194 \times 10^{21}}\, e^{-s/1210}\, s^{5.2}$$

Determination of the Areas Under the Weibull Distribution for Reliability Calculation

The Weibull density function (5, p. 91) is

$$f(s) = \frac{\beta}{\eta}\left(\frac{s - \gamma}{\eta}\right)^{\beta - 1} e^{-\left(\frac{s - \gamma}{\eta}\right)^{\beta}} \qquad (2.3.9)$$

The partial area under this distribution's density function is given by

$$F = \int_0^{s_1} f(s)\,ds \qquad (5.4.24)$$

or

$$F = 1 - e^{-\left(\frac{s_1 - \gamma}{\eta}\right)^{\beta}} \qquad (5.4.25)$$

Substitution of values of $s_1 > \gamma$, β and η in Equation (5.4.25) gives the value of F. The corresponding value of G may be calculated similarly. This would then enable the calculation of the component reliability by the Transform or Digital Computer method.

CHAPTER 5.5

COMPUTER SOLUTION OF THE GENERAL RELIABILITY EXPRESSION

The general reliability expression

$$R = \int_{-\infty}^{\infty} \int_{s}^{\infty} f(s)\, f(S)\, dS\, ds \qquad (5.3.4)$$

may be evaluated by numerical methods of integration adapted to digital computers for expediency. One such method uses Simpson's Rule (6, p. 348).

The method consists of applying the rule to the inner integral over finite, arbitrary intervals of the variable of integration until the evaluation reaches the desired degree of accuracy and then reapplying the rule for the outer integral whose integrand now consists of the product of the outer function and the terms in the result of the first evaluation. This second evaluation is in turn carried out until the final degree of accuracy is obtained. The size of the intervals over which the evaluation is made will depend on the error that can be tolerated and the form of the function involved. The infinite limits of the integrals must be truncated at finite endpoints which are located far enough in either direction to keep any area contribution outside these endpoints at a negligible level.

Simpson's Rule is

$$\int_{a}^{b} f(x)dx \cong \frac{b-a}{3n} \Big[f(x_0) + 4f(x_1)$$

$$+ 2f(x_2) + \ldots + 2f(x_n - 2)$$

$$+ 4f(x_n - 1) + f(x_n) \Big] \qquad (5.5.1)$$

Where n is the desired number of uniform intervals, as shown in Fig. 5.3, i.e., $\Delta x = \frac{b - a}{n}$, $f(x_n) = f(b)$, and $f(x_o) = f(a)$.

For any n, an approximation for $\int_a^b f(x)dx$ can be obtained with an error,

$$E \leq \frac{(b - a)^5}{180n^4} K$$

where K is the maximum value $|f^4(x)|$ attains for x in interval $a \leq x \leq b$. The value of the first approximation, I_1, is subtracted from that of the second approximation, I_2, and the result is compared with the desired accuracy, δ. This comparison is continued until $I_k - I_{k-1} \leq \delta$. Then I_k is the desired approximation for $\int_a^b f(x)dx$. Here a and b are the truncated limits of the distribution.

A similar procedure is used for the outer integral until the desired accuracy is obtained either by extending the truncation limits or the number of intervals or both. In this manner the reliability of a component having any type of a continuous distribution of stress and strength can be determined. A digital computer program for this method is available at The University of Arizona.

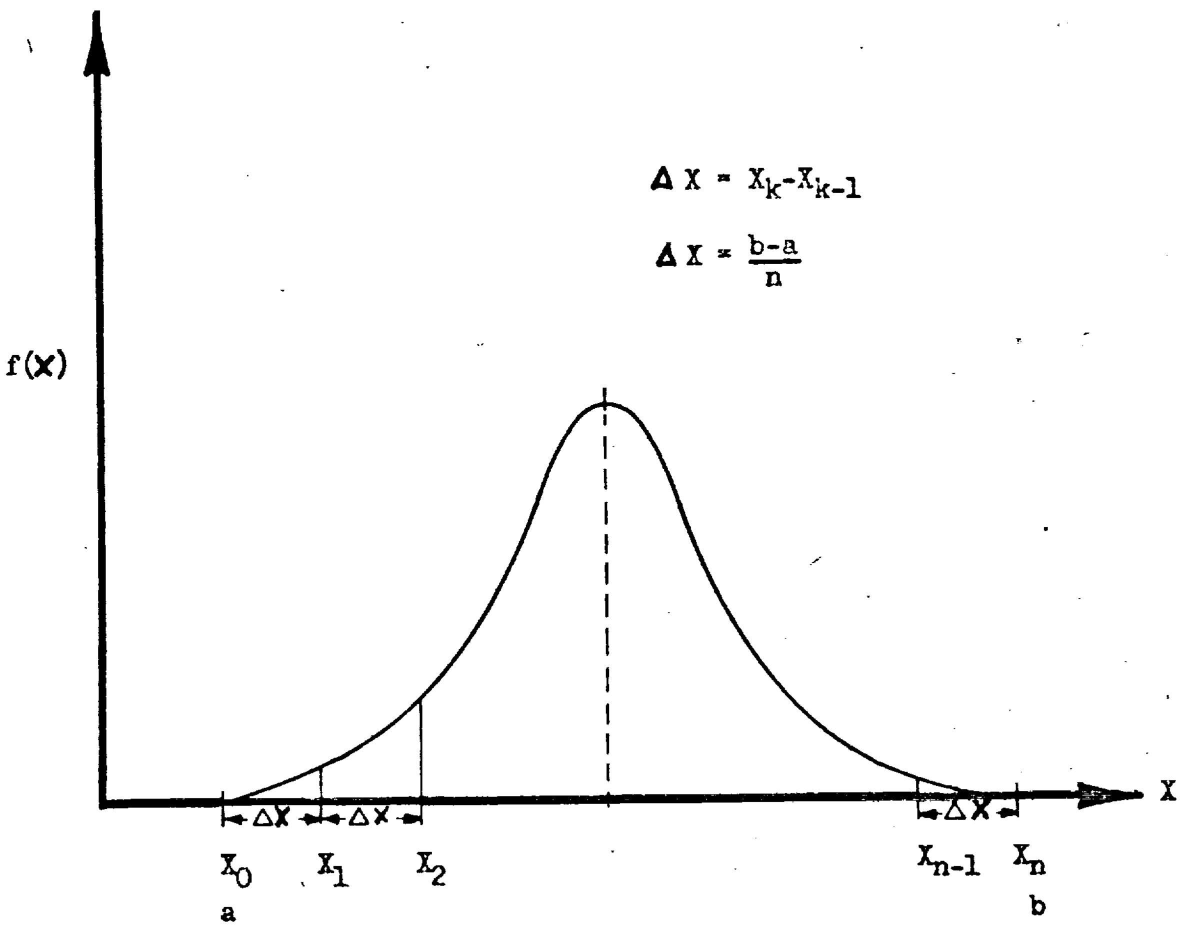


FIGURE 5.3 INTERVAL AND VARIABLE DESIGNATION FOR SIMPSON'S RULE OF NUMERICAL INTEGRATION

CHAPTER 5.6

CONCLUSIONS AND RECOMMENDATIONS

The methodology presented in this section allows the calculation of the reliability of components, given their failure governing stress and strength distributions. These distributions may be normal, lognormal, gamma, Weibull, or any other continuous probability density functions. Analytical, as well as transform and digital computer solutions have been provided to calculate reliabilities.

The knowledge of how to calculate the component reliability provides the necessary tools needed to design a specified reliability directly into a component. In other words, the direction of changes that need to be made to the means, standard deviations, and other stress and strength distribution parameters can be established and various values of these parameters can be tried until the desired reliability is obtained. Specific design and cost considerations would dictate what changes should be preferred. This is a fine case of a computer application whereby these parameters can be quickly optimized to obtain the stress and strength distributions required for a specified reliability, at minimum cost.

It is recommended that engineers, and in particular design engineers, use these methodologies and also contribute to their refinement since the advent of computers has made the seemingly unbearable labor involved shrink swiftly to highly satisfactory proportions. Furthermore, it is recommended that the designer think in terms of reliabilities, i.e., probabilities of success and unreliabilities, i.e., probabilities of failure, rather than in terms of the safety factor or the safety margin alone.

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SECTION 6

THE UNIVERSITY OF ARIZONA'S COMBINED BENDING AND TORSION FATIGUE TESTING MACHINE FOR RELIABILITY RESEARCH

CHAPTER 6.1

INTRODUCTION

The previous sections have provided the theoretical approaches to determine the reliability of mechanical parts by the design-by-reliability methodology. It was pointed out that much statistical design data on components subjected to complex fatigue made of materials being presently used in the NASA space effort need to be generated. The ultimate objective is to fill the gap in the prediction of the reliability of systems where purely mechanical components are involved. The great need for this effort is exemplified by statements to be found in numerous reliability specifications, and books (1 through 29). The methodology has the goal of determining the reliability of mechanical components once the failure-governing stress and strength distributions are known. As discussed in Section 5, and indicated in Figure 6.1, the unreliability and then the reliability can be determined.

The component of concern in this research is a specimen which simulates a shaft in service (30, 31). The specimen material for which data needs to be generated is SAE 4340 steel, Cond. C. as per MIL-S-5000B (32). The loading consists of reversed bending and steady torque applied to a rotating specimen with a stress concentration, which produces combined-stress, or complex, fatigue. It is to meet this data need that special fatigue testing machines capable of providing these test conditions had to be designed and built.

The following chapters give the details of the design of these machines.

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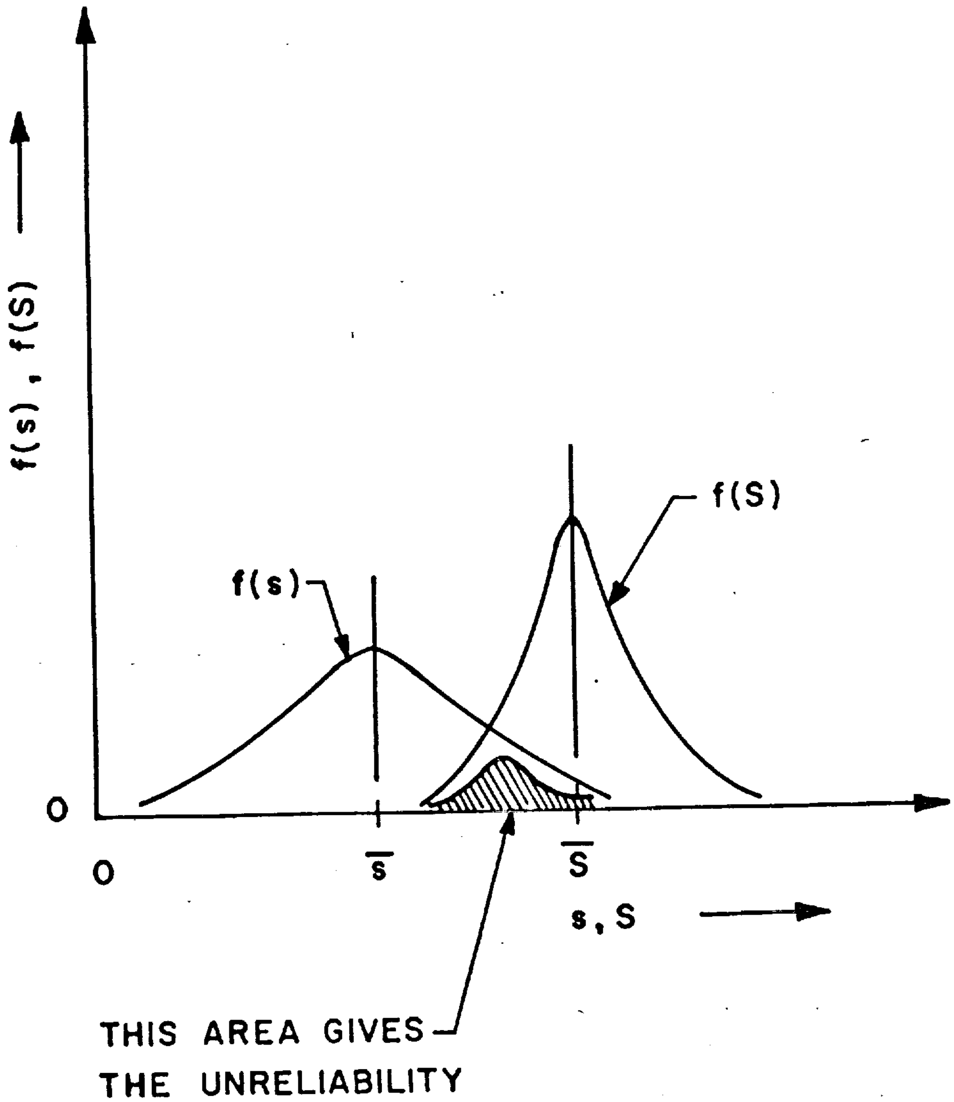

FIGURE 6.1 STRESS - STRENGTH INTERFERENCE ZONE

CHAPTER 6.2

[illegible] FOR THE TESTING MACHINE

Objective

The main objective of the effort described in this section was to build a fatigue testing machine that would generate failure data for design by reliability techniques by subjecting specimens to a steady torque and a reversed bending moment.

Specifications

The design of the fatigue testing machine was based on five criteria: (1) have a high test rpm for fast generation of failure data, (2) use readily available commercial components, (3) provide loading means to develop a steady torque and a reversed bending moment, (4) accommodate cylindrical test specimens and (5) accept the required instrumentation.

"Off-the-shelf" components were located to perform the specified functions whenever this was possible. Special orders were avoided and locally manufactured parts were kept to a minimum.

Testing speed of specimen.-The time required to fail a specimen is a function of the revolutions per minute of the test machine. The higher the rpm the less time required for failure of test specimens and the faster the life distributions can be generated at a specified stress level. For example, near the endurance limit of a material, a greater amount of testing time is required, since 10^6 or more cycles must be accumulated.

The speeds that can be obtained for testing purposes, without using variable speed motors, are 1200 rpm, 1800 rpm, and 3600 rpm. The higher rpm, 3600, is desired but 1800 rpm would be acceptable. The lower rpm, 1200, was considered slow and the cost of this motor is more than the 1800 rpm motor.

Required loading mechanism.-A torsional device capable of producing a sufficient amount of torsional stress to fail the specimen in static loading was used. The apparatus for the required bending moment was designed to produce a uniform bending moment.

The machine was designed for 5400 in-lb of steady torque and 3540 in-lb of reversed bending moment. The design loads were obtained by using a combined load factor of two.

Test specimen.-NASA's Lewis Research Center specified the requirements of the test specimen (30). Also the test specimen has to be designed in conjunction with the SNAP VIII turbine shaft, Aerojet-General D/N 091273, sheets one and two (31). Sheet one of the drawing supplied the fabrication processes of the turbine shaft. The test specimen was not as elaborate as the turbine shaft, but the processes were closely followed. The applicable processes are covered in Table 6.1.

The largest test specimen that the machine was designed for could have a major diameter, D, of 1.5 in. Because of the specified material, SAE 4340, a D of approximately 0.750 in, and a minor diameter, d, of 0.500 in. was selected. A groove was included to incorporate a stress concentration factor. The radius of the groove was selected as 0.150 in., giving a SCF of approximately 1.5, which correlates with an area of high stress concentration on the SNAP VIII shaft, as indicated in Table 6.1. The American Society for Testing Materials, Manual on Fatigue Testing (33) was also adhered to.

Instrumentation. - The major instrumentation consisted of (1) strain gages for static and dynamic strain measurements, (2) slip rings to transmit the desired information off of the rotating machinery to a stationary recorder, (3) amplifiers, and (4) a recorder to permanently record the strain gage outputs.

Table 6.1 gives a resume of the design requirements that were followed during the project as set forth by NASA and pertinent correspondence.

TABLE 6.1

RESUME OF DESIGN CRITERIA

Components	
Test Machine	Rotating specimen, 3600 rpm desired, 1800 rpm acceptable; produce and hold steady torque and reversed bending moment; holding chuck-1.5 in. diameter maximum; simple design employing "off-the-shelf" components.
Loading Mechansim- Steady torque	Simple device to produce, hold, and transmit desired steady torque of 5400 in-lb to test specimen.
Loading Mechanism- Reversed bending moment	Simple device to produce a reversed bending moment of 3450 in-lb while specimen is rotating.
Test Specimen (See Figure 6.7)	SAE 4340 condition C-4, MIL-S-5000B, certification of chemical and physical properties, uniform quality, same heat and processing, heat treat Rockwell "C" 35/40 as per MIL-H-6875 with minimum tempering temperature of 1000° F., inspection as per MIL-I-6868. $D = 0.735$ in. $d = 0.500$ in. $r = 0.150$ in. SCF = 1.5
Instrumentation:	
Strain gages	Dynamic and static measurements.
Slip rings	Transfer of strain gage data to amplifier while specimen is rotating.
Amplifier	To handle static and dynamic information.
Recorder	To produce a permanent record of strain gage information.
General	Equipment to handle at least 2 sets of information simultaneously.

CHAPTER 6.3

SELECTION OF TESTING TECHNIQUE

Survey of Commercial Companies

Many commercial companies (34-39) were contacted. These companies neither produced nor possessed a fatigue testing machine with the ability to produce steady torque and reversed bending moment. The Budd Co. (40) suggested that Mabie or Gjesdahl be contacted concerning their fatigue testing machine (41), this machine will be presented later in this chapter. In conclusion, a commercial machine could not be located for this project.

Modifying Existing Machines

The University of Arizona's universal fatigue testing machine.-Based on the negative survey, it was decided to design and build a new testing machine or modify an existing one. Four fatigue testing machines were considered. The first was a modification of The University of Arizona's Universal Fatigue Testing Machine, Model Number SF-01-U-2, so that it could handle the combined-stress fatigue problem. A loading fixture would have had to be designed to couple-in the steady torque part of the desired loading. This fatigue testing machine has a non-rotating specimen. With only one machine available at The University of Arizona, additional testing machines would have to be purchased.

Considering the purchase of additional machines, at a high cost, and the extensive modification, the Universal Testing Machine was considered not acceptable.

Modified R. R. Moore fatigue testing machine.-The second consideration was to modify a commercially available fatigue testing machine. An R. R. Moore High Speed Fatigue Testing Machine (42) would have had to be redesigned to provide the required steady torque. Preliminary work on the proposed modification was done by Joe McKinley (43). This proposal consisted of incorporating an energy dissipating device to provide steady torque, as shown in Figure 6.2. The dissipation of the energy developed from the steady torque requirement was very high in the above machine. It was decided that a simpler technique of developing steady torque should be employed for this project. The required modification for steady torque and the buying of the expensive R. R. Moore Testing Machines was not desirable for the NASA project.

Hughes fatigue testing machine.-R. Hughes (44) while working with Dr. D. B. Kececioglu designed a fatigue testing machine shown in Figure 6.3. The steady torque is developed by a hydraulic pump. A test specimen is placed between two bearings with the load being applied on one end. This load for reversed bending moment is applied by varying the gearing ratios between two shafts.

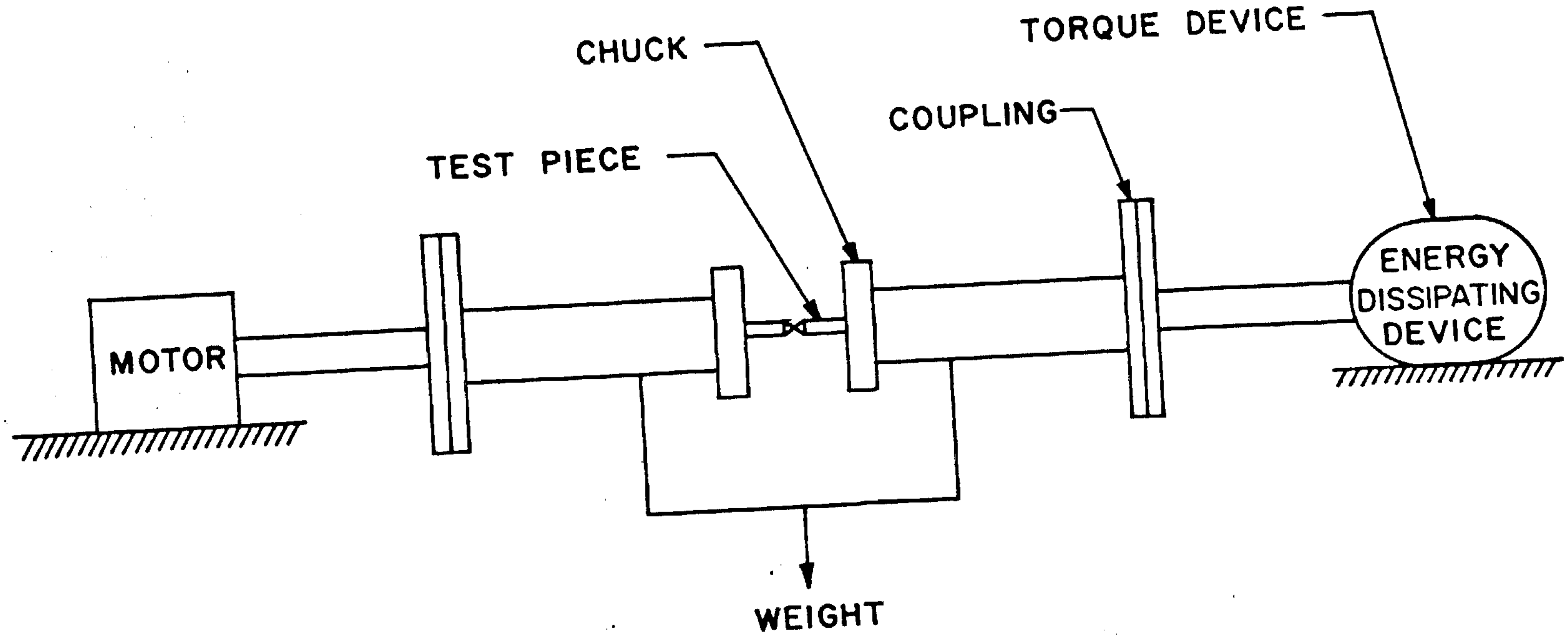

FIGURE 6.2 PROPOSED MODIFICATION OF R.R. MOORE TESTING MACHINE

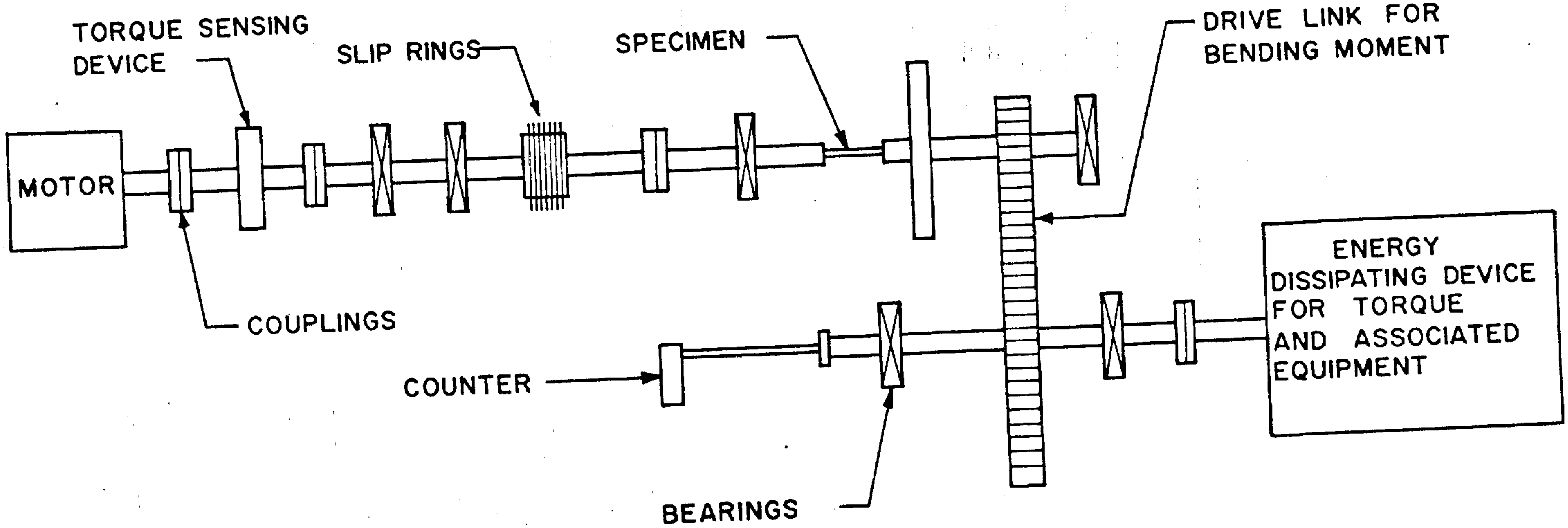

FIGURE 6.3 PROPOSED HUGHES FATIGUE TESTING MACHINE

Actual service conditions are closely approximated by the Hughes Machine. However, the machine has many intricate parts, it is complicated to build, and required an energy dissipating device for steady torque. Therefore, it was not selecte as the type of testing machine required for the project.

Foster fatigue testing machine.-The fourth machine began with a suggestion by Professor A. G. Foster (45), who indicated that the test specimen could be held in a non-revolving position with the required bending loads rotating. The steady torque requirement could be applied externally and varied as required. The principle used in this fatigue testing machine has not been found in the literature survey, and is thought to be a new concept. The bending moment is obtained by applying a bending load to the free end of the test specimen through a double eccentric. To obtain the reversed portion of the bending moment the free end of the specimen is then moved in a circular path. The effect of holding the specimen stationary and rotating the forces is not certain. It is possible that statistically significant differences in results may be found when the specimen is stationary as compared to its being rotated. Therefore, it was felt that the machine could not be used for the present project. Figure 6.4 is a sketch of the apparatus. Not included is the complete structure for holding one end and eccentrically oscillating the other end.

Literature Survey

A literature survey* was made in order to locate fatigue testing machines to generate the desired failure data. "References on Fatigue" (46) was surveyed from 1955 to 1963. The only paper of interest was the Symposium on Large Fatigue Testing Machines and Their Results, (47). No testing machines capable of handling combined steady torque and reversed bending moment were found in the paper. Other references (48, 49) were reviewed; information concerning combined-stress fatigue machines was not found.

The Proceedings of the Society for Experimental Stress Analysis (50) from 1945 to 1960, and "Experimental Mechanics" (51) from 1961 to December 1965, were reviewed in an attempt to locate a combined steady torque and reversed bending moment testing machine. Several fatigue testing machines were found, but only one was of direct interest to the NASA contract; a testing machine built by Mabie and Gjesdahl (41). This machine uses the four-square principle for applying the steady torque while the rotating beam principle is used to produce the bending moment.

The four-square principle is not a new principle for developing steady torque. Industrial corporations, such as gear manufacturers, speed reducer manufacturers, and coupling manufacturers all use this principle to evaluate their products (52).

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*Refer to Section 8 in the BIBLIOGRAPHY at the end of this report for a discussion of the testing machines which were found.

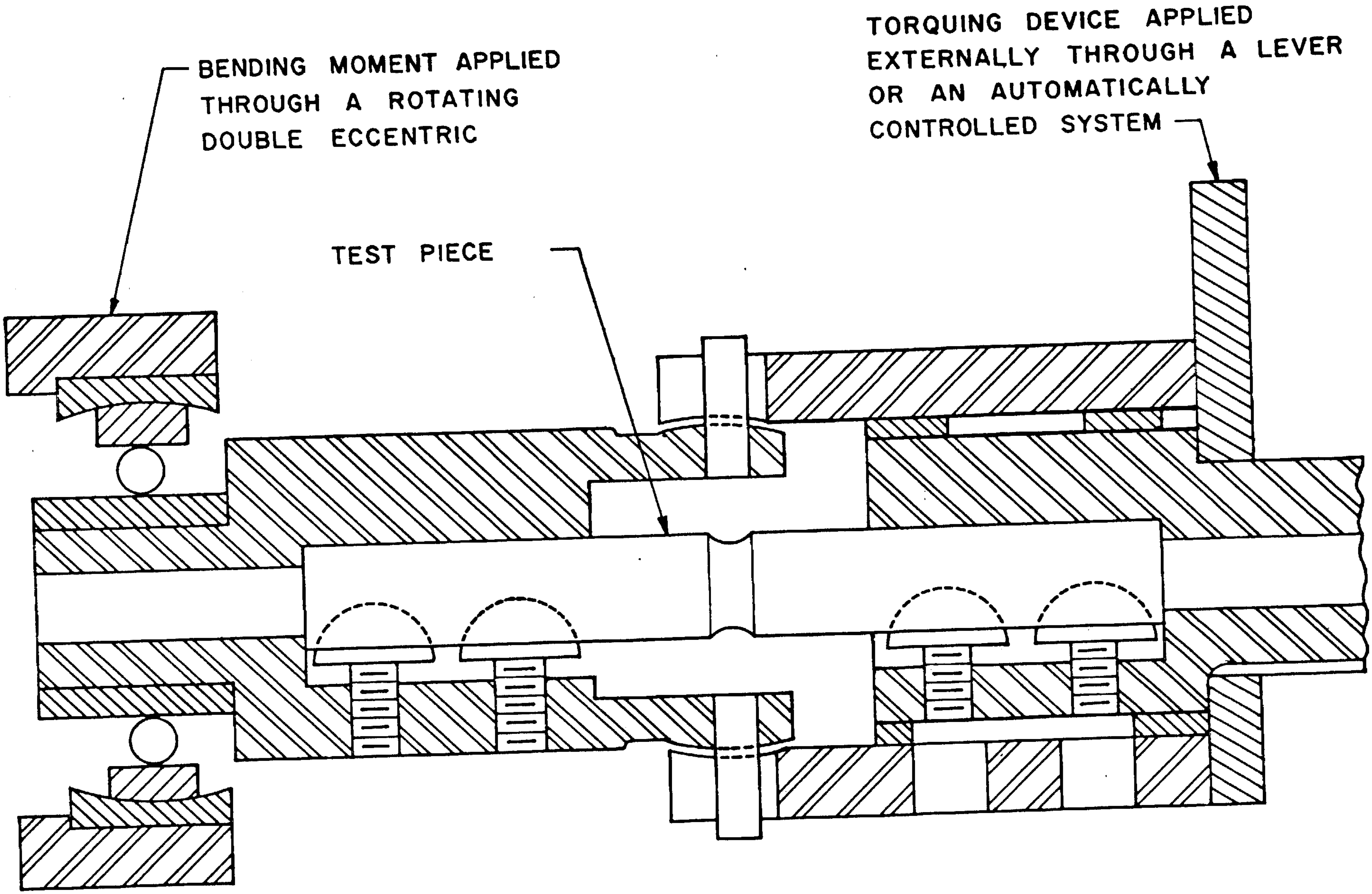


FIGURE 6.4 FOSTER FATIGUE TESTING MACHINE

In the Mabie-Gjesdahl Machine this principle was used to develop a maximum steady torque of 6000 in-lb, however, the machine was only operated at a maximum of about 2000 in-lb of torque (41, p. 86). At this loading the machine produced a high pitch whine (53), a result of the pitch line velocity of the spur gears being 3000 to 4000 feet per minute (54). Also the pre-set torque could not be maintained. The steady torque, four-square principle, is coupled with a reversed bending moment, as shown in Figure 6.5.

The desired bending moment was applied to the test piece so as to simulate a simply supported beam. Through the use of a hydraulic cylinder and associated equipment the required bending moment load was developed (53). It appears that a reduction in bending moment might occur during the testing phase as a result of the hydraulic cylinder leakage. The bending moment is constant along the length of the test piece for a specific value of the bending load. The machine was designed for 5000 in-lb and operated at a maximum of about 3200 in-lb of bending moment. The reversed bending moment was gained through the rotation of the test piece in the four-square mechanism.

The Mabie-Gjesdahl test machine operated at 1200 rpm. The machine was driven by a 3 hp, 1200 rpm induction motor (55). Mabie (53) furnished two assembly drawings (56, 57) and additional design information as to the problem areas of his test machine, as shown in Table 6.2.

The exact instrumentation on the Mabie-Gjesdahl test machine is not known. However, the torque values are measured and checked only in a static situation. The bending moment values were checked and related to the pressure gage on the hydraulic equipment. The load was applied statically and the pressure noted. Strain gages were used for the static torque measurements and also for the bending load. The bending load strain gages were mounted on the loading bar.

The test machine was calibrated dynamically with suitable mounted strain gages and slip-ring and brush assemblies. The exact equipment is not known. Correlation of these dynamic tests were made to the stresses obtained through calculations and an 8-12% error was noted (41).

The four-square principle was selected as the means of applying the steady torque and the rotating beam principle was selected for applying the reversed bending moment. Also this principle combined with the rotating beam principle for the bending moment is a proven principle for fatigue testing. The major advantage of the four-square principle is that the required horsepower for the loading of the specimen is internal. Therefore, the driving mechanism only has to overcome the frictional losses of the machine.

Correspondence with Mabie (58) indicated that the commercially purchaseable components exceeded $5000.00. With these thoughts in mind a test machines similar to the Mabie-Gjesdahl principle was designed and built at The University of Arizona.

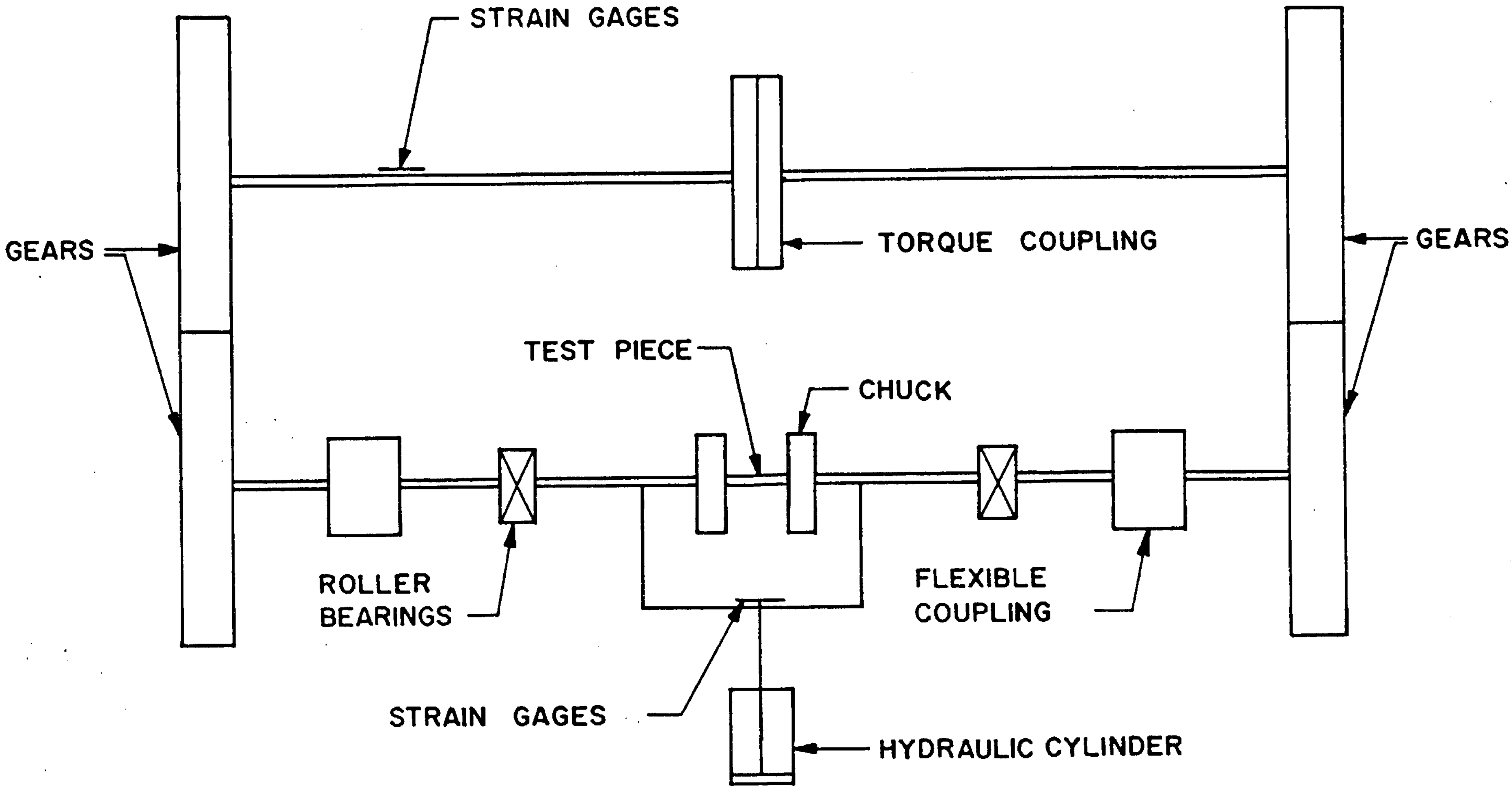


FIGURE 6.5 MABIE – GJESDAHL FATIGUE TESTING MACHINE

TABLE 6.2

COMPARISON OF TESTING MACHINES

Testing Machine	Advantages	Disadvantages	Remarks
Mabie-Gjesdahl	Does not dissipate energy to apply torque to specimen. The machine has been built and is functional. Assembly drawings available. Uses 1" diameter specimen.	Difficult to hold constant torque. Vibration present. Noisy.	Cost $5000.00 for commercial parts. Proven principle. Speed = 1200 rpm. Press fit on holding chucks. Gear box - spur gears.
Hughes	Comes closest to simulating actual service conditions. Preliminary drawings and design nearly completed. Parts list nearly completed.	Many intricate parts. Complicated to build. Much vibration and noise is possible at 1800 rpm or greater. Dissipates energy to torque the specimen. 1/4" diameter maximum specimen.	Not chosen for project. Application of bending moment requires several gear ratios.
Foster	Adaptable to any turret lathe. Capable of high rpm.	Not a proven machine. Automatic shut down may be difficult. Torque may not be steady.	Requires much research and development. Not chosen for project.
Modified R. R. Moore or Locally Manufactured Version	Will run smoother at 3600 rpm. Locally manufactured version lower in cost than starting with the R. R. Moore Machine	Uses large motor. Dissipates large amounts of energy. Not a proven machine. 1/4" diameter specimen maximum.	Expensive machine. Not chosen for project.
Modified Sonntag Fatigue	Available now. Slip rings not required for dynamic measurements.	Must be modified for combined stress fatigue. Fixture for steady torque must be manufactured. Not proven for combined loads of the desired type.	Expensive machine. Not chosen for project.

CHAPTER 6.4

COMBINED BENDING AND TORSION FATIGUE MACHINE

Initial Design of Test Machine

Summary of initial design.-The initial design of the test machine was based on the amount of torque and bending moment required to completely fracture a test specimen with a minor diameter of 0.700 in. that rotated at 3600 rpm.

The required static torque was 7470 in-lb at 10^3 cycles, and the required bending moment was 4500 in-lb for complete fracture of the SAE 4340 specimen for the worse case condition. The machine was then analyzed with a load factor of two. The initial design loads were 9000 in-lb in bending and 15 000 in-lb in torque.
Since,

$$hp = \frac{Tn}{63\ 000} \qquad (59,\ p.\ 55) \qquad (6.4.1)$$

Substituting,

$$hp = \frac{(15\ 000)(3600)}{63\ 000}$$

$$hp = 847$$

The internal horsepower, the horsepower that the specimen sees, was a large value for equipment operation. The gear box capable of sustaining 847 hp would have weighed 1910 lbs with physical dimensions of 19 in. wide by 43 in. deep by 29 in. high. Since two boxes are required the total weight, not including auxilary components, was about 4000 lbs.

Another difficulty was that the bearings for the loading frame would have had to be operated in an oil bath system to allow proper lubrication at 3600 rpm.

Design recommendations.-It was realized that design changes would have to be made to obtain a more desirable fatigue test machine. Lowering the internal horsepower was a necessity. Equation (6.4.1) indicates that hp decreases as T and n decrease. The torque is a function of the diameter of the specimen, therefore by reducing the diameter of the specimen the amount of torque for complete fracture can be reduced, as may be seen from

$$T = s_m \frac{J}{c} = s_m \frac{\pi d^3}{16} \qquad (6.4.2)$$

A diameter, d, of 0.5 in. was chosen which reduced the torque by a factor of 2.82.

Also a reduction of the rpm to 1800 provided a more realistic selection of some of the components. The main result of the reduction of rpm and T was that it allowed smaller and lighter components to be selected. A complete analysis of the fatigue testing machine follows.

The University of Arizona Fatigue Testing Machine

The basic changes in the design specifications of the test machine were to reduce the speed to 1800 rpm and the specimen's neck diameter to 0.5 in. Each component was studied starting from the test specimen and proceeding around to the back shaft. The design of the components follows.

Test specimen (See Figure 6.7a):

Strength of Specimen in Reversed Bending Only

$$S_e \text{ @ } 10^3 \text{ cycles} \cong 0.9\ S_{UT} \tag{6.4.3}$$

$$S_u = 160 \text{ ksi} \quad (60, \text{ code } 1206, \text{ p. } 1)$$

Therefore,

$$S_{e_{10^3}} \cong 0.9\ (160)$$

$$S_{e_{10^3}} \cong 144 \text{ ksi}$$

$$s_a = S_{e_{10^3}} = \frac{M}{I/c} \quad (59, \text{ p. } 100) \tag{6.4.4}$$

Rearranging,

$$M = \frac{s_a I}{c} \ * \tag{6.4.5}$$

$$= \frac{(s_a)(\pi\, d^4/64)}{(d/2)}$$

$$= \frac{(144 \times 10^3)(\pi)(0.5)^3}{32}$$

$$M = 1770 \text{ in-lb}$$

---

* The stress concentration factor has been omitted from this calculation in order to provide a conservative analysis, so that the testing of solid 1/2 inch diameter specimens or of larger diameter specimens but with higher stress concentration factors will be within the capability of the machine.

Strength of Specimen in Steady Torque Only

Strength in steady torque will equal the torsional modulus of rupture, or

$$S_s \Big| = 110 \text{ ksi} \quad (60, \text{ code } 1206, \text{ p. } 9)$$

$$s_m = \frac{Tc}{J} \quad (59, \text{ p. } 52) \qquad (6.4.6)$$

$$T = \frac{s_m J}{c}$$

$$= \frac{(s_m)(\pi\, d^4/32)}{(d/2)}$$

$$= \frac{(110 \times 10^3)(\pi)(0.125)}{16}$$

$$T = 2700 \text{ in.-lb.}$$

These values represent the worst case to fracture the specimen.

Load Factors

A load factor equaling two was used to establish the design loads for the fatigue test machine: for potential increase in material strength $K_1 = 1.5$ and for contingencies $K_2 = 1.33$. Then the combined load factor is

$$K_1 K_2 = (1.5)(1.33)$$

$$K_1 K_2 = 2$$

Load factor = 2

The design loads for the test machine are 3540 in - lb for bending moment or 2 x 1770 and 5400 in - lb for torque or 2 x 2700.

Stress Concentration Factor

A stress concentration was incorporated into the test specimen, details of which are given in D/H UANASA-6700-B-002. This stress concentration is to develop a SCF approximating that in the SNAP VIII turbine shaft. **(See Figure 6.7).**

The SCF is obtained as follows. (See Table 6.1 and Figure 6.7):

$$d/D = \frac{0.500}{0.730}$$

$$d/D = 0.685$$

$$r/D = \frac{0.145}{0.730}$$

$$r/D = 0.198$$

Therefore,

$$K_t = 1.45^{*} \quad (61, \text{ p. } 49) \qquad (6.4.7)$$

<u>Endurance Limit</u>

$$S_e = k_a\, k_b\, k_c\, k_d\, k_e\, k_f\, S_E' \quad (62, \text{ p. } 166) \qquad (6.4.8)$$

Refer to D/N UANASA-6700-B-002.

The modifying factors are:

Surface finish: $k_a = 0.89$ for ground finish (62, p. 167)

Size: $k_b = 0.85$ for $D < 2.0$ in. bending (62, p. 168)

Reliability: $k_c = 1$, because it will not be handled this way.

Temperature: $k_d = 1$, no temperature effect.

Stress concentration: $k_e = \frac{1}{K_f}$ (62, p. 170) (6.4.9)

$$K_f = 1 + q(K_t - 1) \quad (62, \text{ p. } 170) \qquad (6.4.10)$$

$$q = 0.92 \quad (62, \text{ p. } 171)$$

$$K_t = 1.45 \quad \text{from Equation (6.4.7)}$$

* This value of K_t may be as low as 1.41, giving $k_e = 0.725$ and $S_e = 43.8$ kpsi for the respective values calculated on p. 212.

Substitution into Equation (6.4.10), gives

$$K_f = 1 + 0.92\,(1.45 - 1)$$

$$K_f = 1.42$$

Substitution into Equation (6.4.9), gives

$$k_e = \frac{1}{1.42}$$

$$k_e = 0.705$$

Miscellaneous effects: $k_f = 1$

Theoretical endurance limit: $S'_E = 0.5\,S_u$ (62, p. 162) (6.4.11)

$$S_u = 160 \text{ ksi} \quad (60, \text{ code } 1206, \text{ p. } 1)$$

Therefore,

$$S'_e = 0.5\,(160)$$

$$S'_e = 80 \text{ ksi}$$

Substitution into Equation (6.4.8), gives

$$S_e = (0.89)(0.85)(1.0)(0.705)(1.0)(80)$$

$$S_e = 42.6 \text{ ksi}$$

With S_e and $S_{e_{10^3}}$ a S-N diagram, Figure 6.6, was plotted.

Holding fixture for specimen. - A collet type holding fixture was desired enabling the test specimen to be installed and removed with relative ease. This is superior to press fitting the specimen into a set of holders. The holding fixture was a Balas Tool Holder part No. S16-3"-C12 with collet part No. C-12.

A keyway was machined on the shank of the tool holder to enable a positive transmission of torque. The tool holder was purchased without the cooling equipment and internal threads. Also the collet was modified, and an attachment link was removed in order to provide space for the test specimen key.

Flexible couplings. - There are three couplings on the test machine: two on the front shaft that are identical and a larger one on the back shaft,

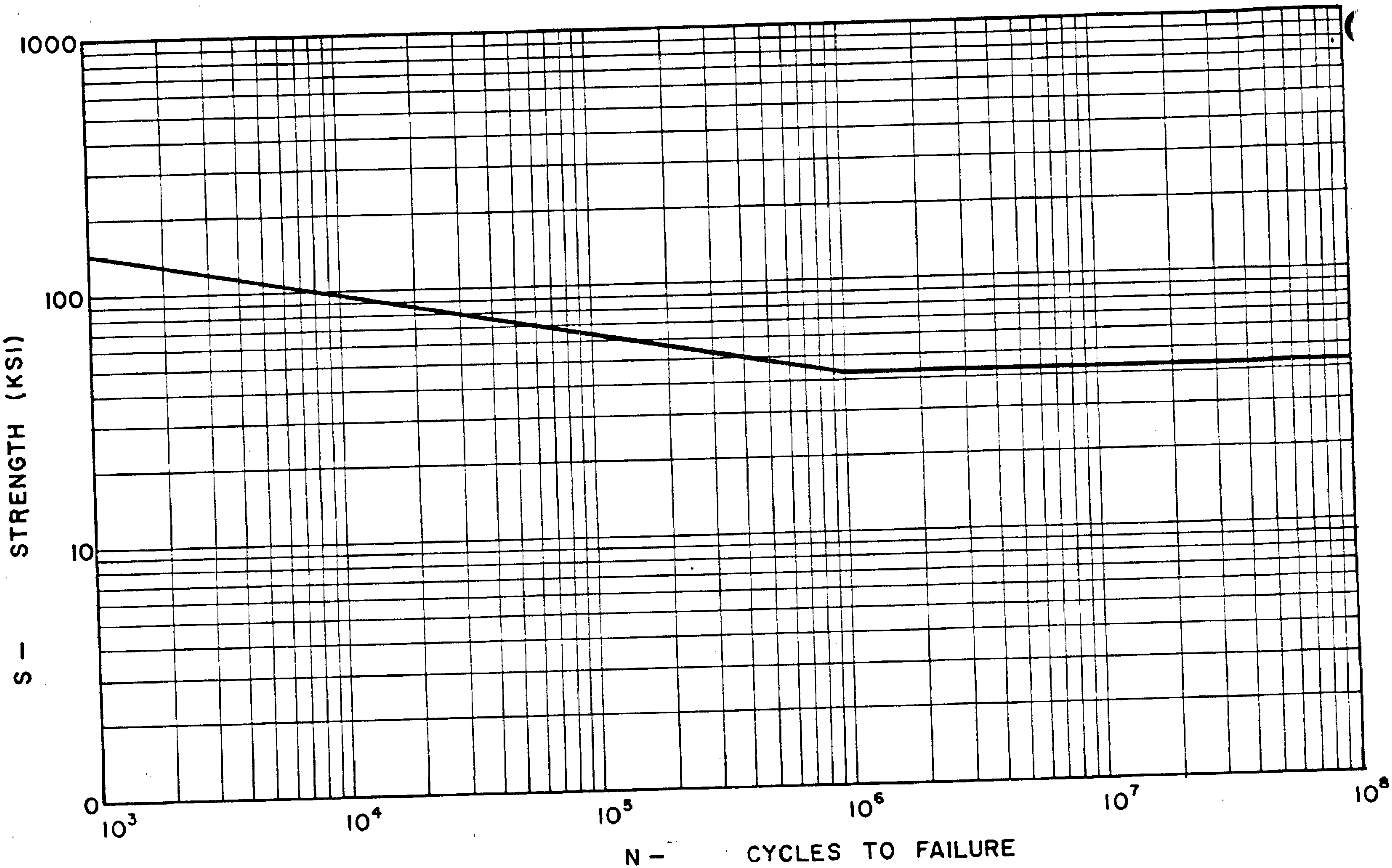


FIGURE 6.6 ESTIMATED S-N DIAGRAM FOR SAE 4340 CONDITION C-4, HEAT TREATED TO R_c 35/40

UANASA - 6700-D-003. The Sier-Bath all steel flexible couplings were selected for the task. Their ability to transmit torque, allow relative movement of shafts, their small size, and relatively low cost, are some of the advantages. The Sier-Bath coupling calculations are presented next. The Link-Belt, Falk, and Rzeppa couplings were also considered. Table 6.3 is a comparison of the analyzed couplings.

High Speed Shaft (1800 rpm)

Sier-Bath Catalog C-5,

$$hp = \frac{Tn}{63\ 000} \tag{6.4.1}$$

$$hp = \frac{(5400)(1800)}{63\ 000}$$

$$hp = 154 \tag{6.4.12}$$

Service factor (SF) = 1.8

$$\text{Rating} = \frac{(hp \times SF \times 100)}{rpm}$$

$$= \frac{(154 \times 1.8 \times 100)}{1800}$$

$$\text{Rating} = \frac{15.4\ hp}{100\ rpm}$$

Select Sier-Bath size 2, rated at 32 hp/100 rpm.

Low Speed Shaft (900 rpm)

$$hp = 154$$

$$SF = 1.8$$

$$\text{Rating} = \frac{(154 \times 1.8 \times 100)}{900}$$

$$\text{Rating} = 30.8\ \frac{hp}{100\ rpm}$$

Select Sier-Bath size 3, rated at 80 hp/100 rpm.

Gear boxes. Normally a one-to-one gear ratio is used in the four-square principle. It was felt that if a commercial gear box could be used much would be gained. For instance, gear boxes provide their own enclosure, they are built by experts in the field, they are time tested, and the parts are readily

TABLE 6.3

COMPARISON OF COUPLINGS

High Speed Shaft

Coupling	Required Rating	Coupling Rating	Safety Factor	Rated RPM	Envelope Dimensions Dia. X L	Weight Lb.	Max. Bore
Sier-Bath No. 2 Std.	15.4	32.0	2.08	18000	4 1/4 X 4 3/4	13	2 1/8
Falk 11FF	21.0	23.0	1.09	3600	8 7/8 X 7 11/16	60	3 9/16
Rzeppa[a] OJ	9.5	30.45	3.2	2500	7 1/8 X 16 5/8	-	-
Low Speed Shaft							
Sier-Bath No. 3 Std.	30.8	80.0	2.6	12 000	5 1/2 X 6 5/8	33	3 1/8
Falk 12 F	35.4	35.0	0.99	3600	9 3/4 X 7 15/16	75	3 7/8

[a]Est. life = 1150 days

available for replacements when failure occurs. Gear boxes from the Western Gear Corporation, Farrell-Birmingham Company, Inc., and the Falk Corporation were analyzed for the test machine. The Falk Corporation gear reducer box was selected. The calculations for all three gear boxes are similar. Only the Falk gear box will be presented. A comparison of all three is provided in Table 6.4.

Gear Box Selection

Service factor = 1.25, Machine tools - Auxilary drive

Equivalent hp = SF x hp

= 1.25 x 154

Equivalent hp = 193

With the preceding value, select Unit 2050 Y 1: single reduction-parallel shaft, 1750 rpm, AGMA gear ratio = 1.84.

Maximum mechanical horsepower = 210

Thermal horsepower without fans = 93 (too low)

Thermal horsepower with fans = 2.5 x 93

Thermal horsepower with fans = 232

Motor Required to Drive Testing Machine

The motor needs to supply only the friction losses in the system.

Gear box loss = 2%/box or	4.00%/test unit
Fan loss = 0.125%/box or	0.25%/test unit
Windage, misc. loss = 0.25%/box or	0.50%/test unit
Total loss	4.75%/test unit

Motor hp required = System hp x Losses

= 154 x 0.0475

Motor hp required = 7.4

Select a 7.5 hp motor.

TABLE 6.4

COMPARISON OF GEAR BOXES

Unit	Required hp	hp Rating Mechanical	Thermal	Speed	AGMA Ratio
Farrel[a] SR - 74	193	173	63	1800	1.52
Western[b] S 53 B	193	209	73	1800	1.837
Falk	193	210	232[c]	1800	1.84

a "Heavy Duty Speed Reducers," Catalogue Bulletin 450 B, Farrel and Birmingham Co., Inc.

b "Speed Master Parallel Shaft-type Gear Reducers," Bulletin 6402, Series D, Western Gear Corporation.

c With cooling fans

Motor Coupling

The appropriate catalog, Falk Bulletin 4100, Dec., 1962, was used. From the machine tool's main drive, the service factor equals two.

Equivalent hp = hp x SF

= 7.5 x 2

Equivalent hp = 15

Select coupling size 5.

Size 5 was not adequate due to maximum bore of 1 1/2 in.. The coupling mounts on the high speed shaft of the 2050 Y1 gear box which has a shaft diameter of 1 3/4 in. Therefore, select the 6 F coupling.

Method of torquing. - The torquing device selected was the Harmonic Drive, Infinit-Indexer, HDUI Series. The largest indexer was selected due to the large low speed shaft. The part No. is HDUI - 200.

Electric motor. - There are many available motors that can handle the required horsepower. As calculated for the gear box selection a 7.5 hp is required. The selection requirements include these items: 7.5 hp, 1800 rpm, and 440 V - 3 phase.

A General Electric, induction, squirrel cage, 440 volt, 3 phase, Type K Tri Clad 700 line motor was selected. These motors and starter units were a donation to The University of Arizona's Reliability Program to be used on the Combined Bending and Torsion Fatigue Testing Machine. The motor has a NEMA 213T frame. An appropriate magnetic starter, with push button controls and the proper fuses, was adapted to the motor.

A 440 V motor was selected over a 120 V motor mainly because its entire motor-starter set up was less in cost. The part numbers are indicated in D/N UAMAS-6700-D-003.

Digital Counter Selection. - Several companies were surveyed for an eight digit mechanical counter capable of 1800 rpm. However there was not a single positive reply. Veeder-Root digital mechanical counters came the closest. An eight digit, worm drive, 1500 rpm mechanical counter was selected, part No. is I-111513. The counter has a one-to-one gear ratio.

The mechanical counter is driven from the motor shaft. A 2:1 ratio is utilized to maintain the counter within the design rpm. A positive drive pulley system was designed for the counter system. The Durkee Atwood Catalog, "Positive Drive Belts," was used for the design.

Positive Drive Selection

Driver NEMA 213T, 7.5 hp, 1800 rpm

Step-1 Table 6.1 NEMA Design B

Class of Driver - II

Table 6.3 Line shafts - Class II

$SF = 1.7$

Table 6.5 Continuous operation add 0.2

Final service factor = 1.9

Step-2 Design horsepower of counter

$$hp = \frac{Tn}{63\ 000} \qquad (6.4.1)$$

Max. static torque for digital counter = 1 oz.-in.

$$hp = \frac{(1800)(1/16)}{63\ 000}$$

$$hp = 0.0009$$

$$\text{Design hp} = hp \times SF$$

$$= (0.0009)(1.9)$$

$$\text{Design hp} = 0.00171$$

Step-3 Table 6.1 Use design hp = 1/12

Speed = 1750 rpm

Select pitch = 1/5

Step-4 Speed ratio = 1

It was evident from the horsepower that any positive drive pulley would be adequate for the task.

Selection of Pulley

Driver: 30 XL037

Driven: 60 XL037

Belt: 220 XL037

Any center to center distance was acceptable.

Base frame. The base frame was designed to hold the gear boxes and provide a mount for the loading frame fulcrum, D/N UANASA-6700-D-009. Two steel, wide-flange beams were used for longitudinal support. Two more steel, wide-flange beams were placed at each end to obtain the necessary height for the loading frame. These two beams were bolted as shown in D/N UANASA-6700-D-009.

There are three mounting plates required for the base frame, D/N UANASA-6700-D-010. Two of the plates were used to provide a surface for the gear boxes. The plates are bolted to the gear box and the frame and also welded into position on the frame. The third plate is used for an accessory mount plate in conjunction with the loading frame. The plate provides a mount for the fulcrum unit, restraining brackets for the horizontal link of the loading mechanism, and a slot to limit the amount of "whip" in the loading mechanism that might be generated during start up and when the test specimen breaks. In addition, a restraining bridge was mounted over each toolholder in order to prevent excessive whipping when a specimen breaks.

Loading frame.- The loading frame consists of several locally manufactured parts along with some commercial components. The design was based on the amount of weight that has to be hung on a loading bar to produce the required bending moment. To keep the weight low the lever arm principle was employed. D/N UANASA-6700-D-004 is the assembly drawing for the loading frame. A bending moment of 3540 in.-lb. was required from the loading frame. To locate the forces at the bearing housings the following analysis was performed. The material for design purposes was SAE C1015 with a $S_y = 62\ 000$ psi.

Bearings and housing: Spherical roller bearings with a tapered inside diameter and a maximum of 3° misalignment were chosen. Two bearings were required along with two adapter sleeves. The bearings are SKF selected from service catalog No. 450.

The bearings were pressed into a bearing housing, D/N UANASA-6700-E-006. Grease cups are mounted on each side of the bearing housing with clamps. A gasket is placed between the housing and cup to prevent grease leakage.

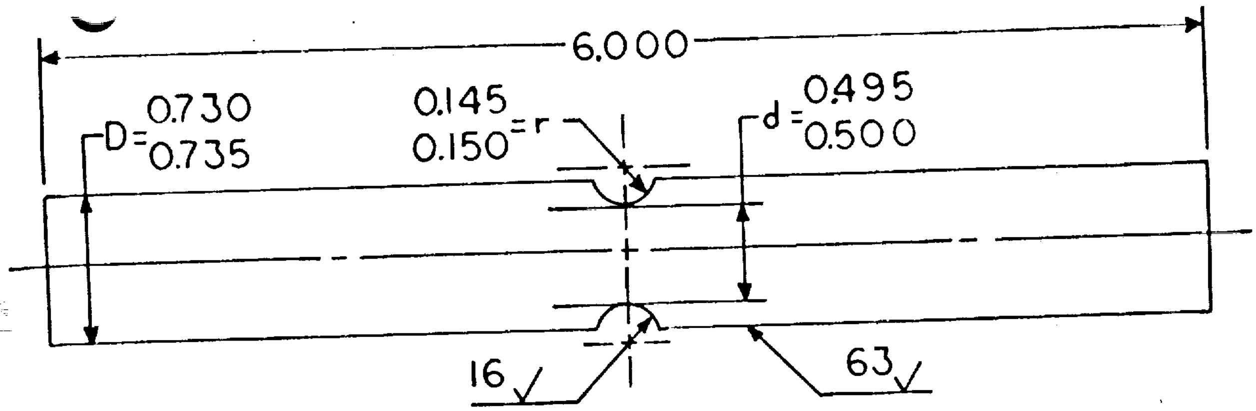
Structural Analysis of Bearing Housing

From Figure 6.7b,

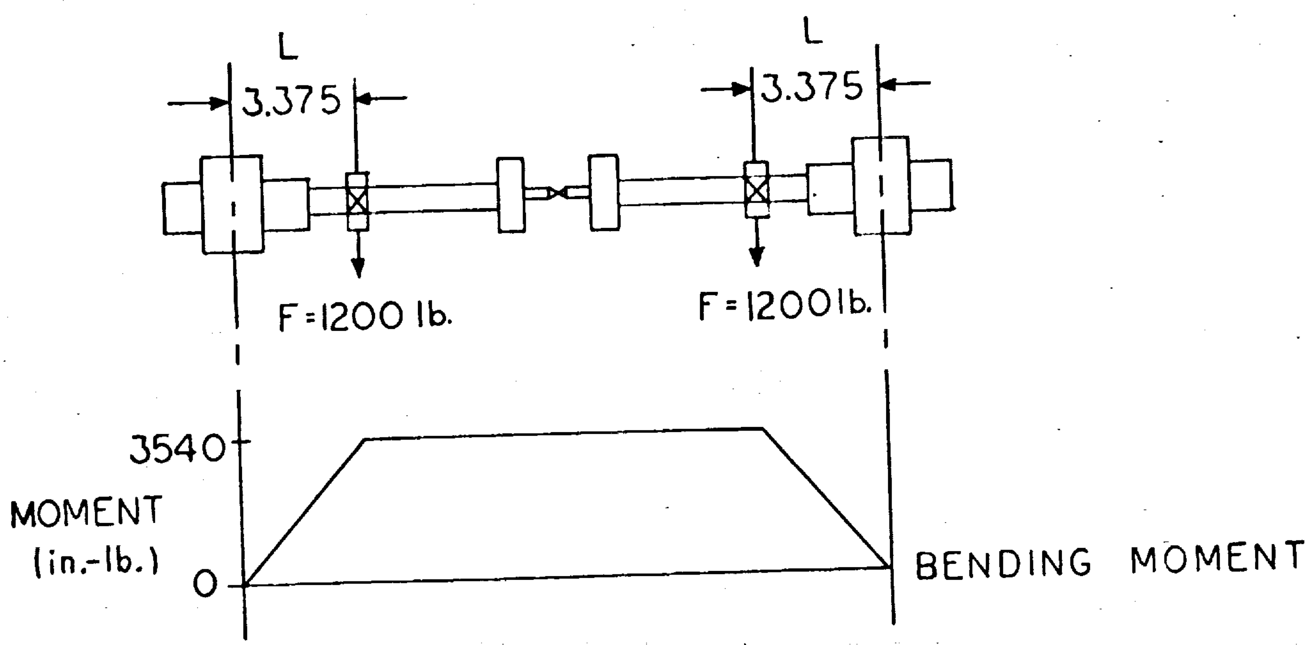
$$M = Fl \tag{6.4.13}$$

$$l = 3.375$$

$$M = 3540 \text{ in-lb}$$



a) UANASA TEST SPECIMEN
6700-B-002



b) LOADING FRAME ANALYSIS

FIGURE 6.7 LOADING FRAME ANALYSIS
AND TEST SPECIMEN

therefore,

$$F = \frac{M}{l} = \frac{3540}{3.375}$$

$F = 1050$ lb. This force will be conservatively taken to be 1200 lb.

From Figure 6.8-b, ring in tension,

$$\sigma = \frac{F}{A} \qquad (6.4.14)$$

$$= \frac{600}{(1)(1/2)}$$

$$\sigma = 1200 \text{ psi}$$

$$\text{Factor of Safety} = \frac{62}{1.2}$$

$$\text{Factor of Safety} = 51.6$$

From Figure 6.8-e, lug in tension,

$$\sigma = \frac{F}{A} = \frac{600}{1(1/2) - (1/4)(1/2)} \qquad (6.4.14)$$

$$\sigma = 1600 \text{ psi}$$

$$\text{Factor of Safety} = \frac{62}{1.6} = 38.8$$

Bolt Analysis

The bolt was a standard hardened steel shoulder bolt that would fasten the horizontal link to the bearing housing.

From Figure 6.9,

$$\tau = \frac{F}{A} \qquad (6.4.15)$$

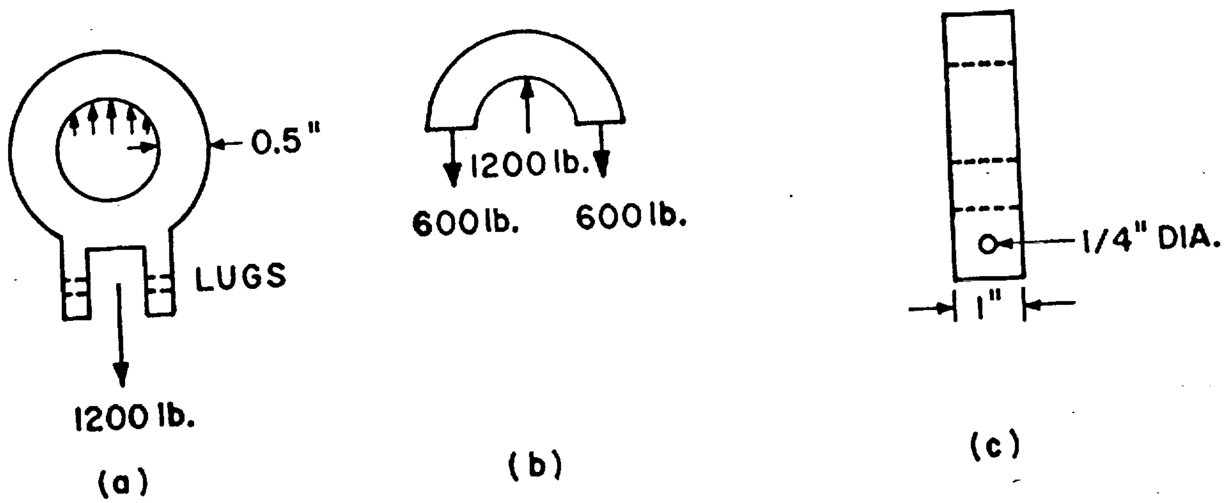
$$= \frac{1200}{0.049}$$

$$\tau = 24\,500 \text{ psi}$$

SAE 2 bolt (63, p. 247) with proof load = 55 000 psi

$$\text{Factor of Safety} = \frac{55}{24.5} = 2.04$$

CIRCULAR SECTION



LUGS

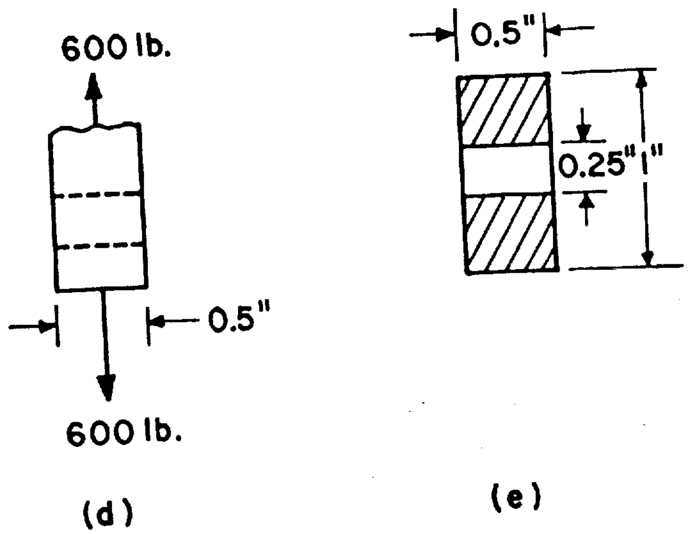


FIGURE 6.8 BEARING HOUSING DIAGRAM

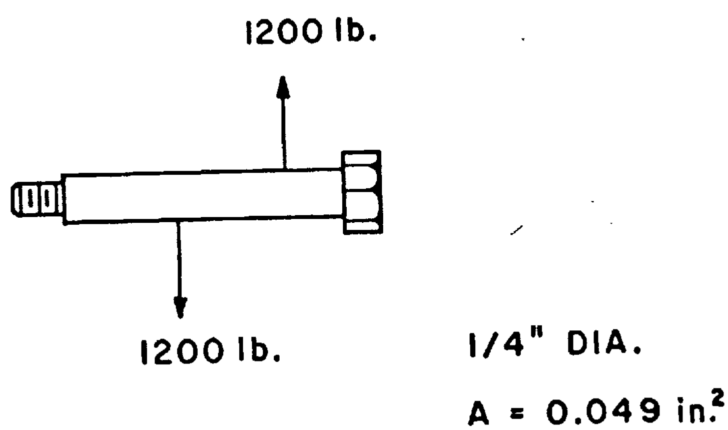


FIGURE 6.9 BEARING HOUSING FASTENER

Horizontal link: The horizontal link allows the required force to be generated from a central point, D/N UANASA-6700-B-013.

Structural Analysis of Horizontal Link

From Figure 6.10-a the point of maximum moment is at the center of the link.

Then,

$$\sigma = \frac{Mc}{I} \qquad c = 3/4'' \qquad (6.4.16)$$

$$M = Fl$$
$$= (8.5)(1200)$$
$$M = 10\ 200 \text{ in-lb}$$

From Figure 6.10-b,

$$I = \frac{1}{12} (b)(h^3_o - h^3_i) \qquad (6.4.17)$$

$$= \frac{1}{12} (1) \left[(\tfrac{3}{2})^3 - (\tfrac{3}{8})^3 \right]$$

$$I = 0.277 \text{ in.}^4$$

$$\sigma = \frac{(10\ 200)(3/4)}{0.277}$$

$$\sigma = 27\ 700 \text{ psi}$$
$$\text{Factor of Safety} = \frac{62}{27.7} = 2.14$$

Vertical links: The forces in the loading frame are transmitted through two vertical links, D/N UANASA-6700-B-014, which are in tension.

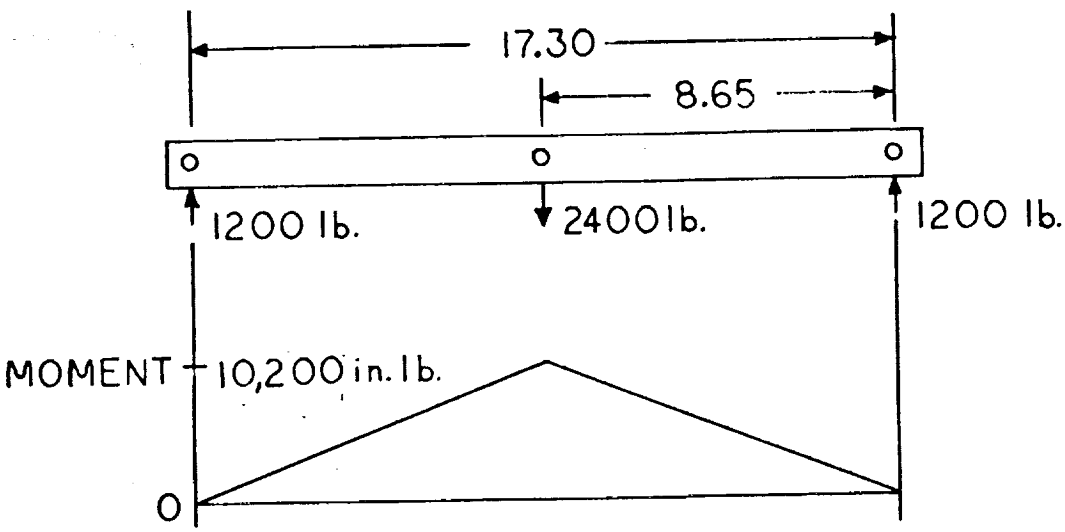
Structural Analysis of Vertical Link

From Figure 6.11-b,

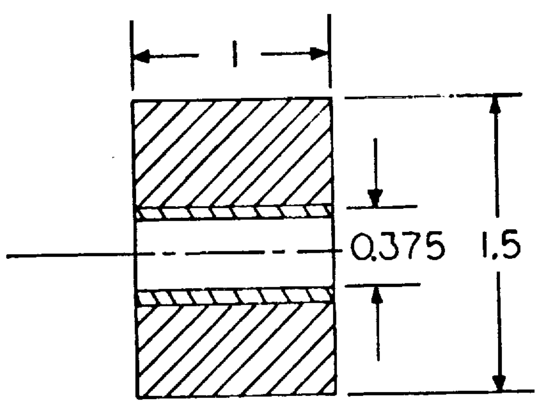
$$\sigma = \frac{F}{A} = \frac{1200}{(1/2)(1)} \qquad (6.4.14)$$

$$\sigma = 2400 \text{ psi}$$
$$\text{Factor of Safety} = \frac{62}{2.4} = 25.8$$

Link bolt: The link bolts are 3/8 diameter SAE grade 2 bolts, with area of 0.755 in.^{2}.

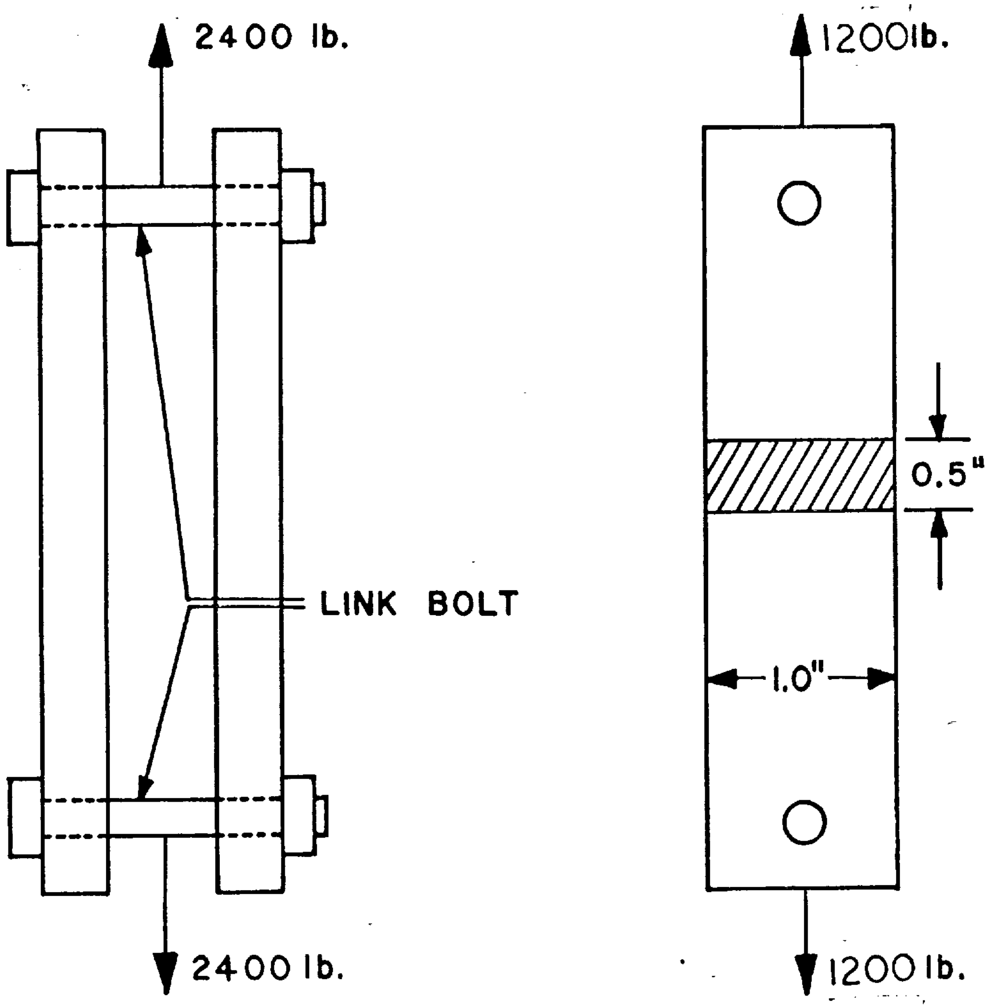


(a) BENDING MOMENT DIAGRAM



(b) SECTION AT MAXIMUM STRESS

FIGURE 6.10 HORIZONTAL LINK DIAGRAM



(a) VERTICAL LINK ASSEMBLY

(b) SECTION - ONE LINK

FIGURE 6.11 VERTICAL LINK DIAGRAM

Structural Analysis of Link Bolt

$$\tau = \frac{F}{A} \qquad (6.4.15)$$

$$\tau = \frac{1200}{0.1105} \qquad (6.4.18)$$

$$\tau = \boxed{10{,}850 \text{ psi}} \qquad (6.4.19)$$

The strength of the bolt is given as 55 000 psi (63, p. 247)

$$\text{Factor of Safety} = \frac{55}{10.85} = 5.08$$

Loading link: The loading link was used as the lever to hang weights to produce the required bending moment. One end of the loading link, D/N UANASA-6700-B-016, has a receptacle to hold the fulcrum block, D/N UANASA-6700-A-017. The loading link was designed as follows:

From Figure 6.12-a,

$$\Sigma M_1 = 0 \qquad (6.4.20)$$

$$2(2400) - 38\,W = 0$$

$$W = \frac{4800}{38}$$

$$W = 126 \text{ lb.}$$

$$\Sigma F = 0 \qquad (6.4.21)$$

$$R_1 + W = 2400$$

$$R_1 = 2400 - 126$$

$$R_1 = 2274 \text{ lb.}$$

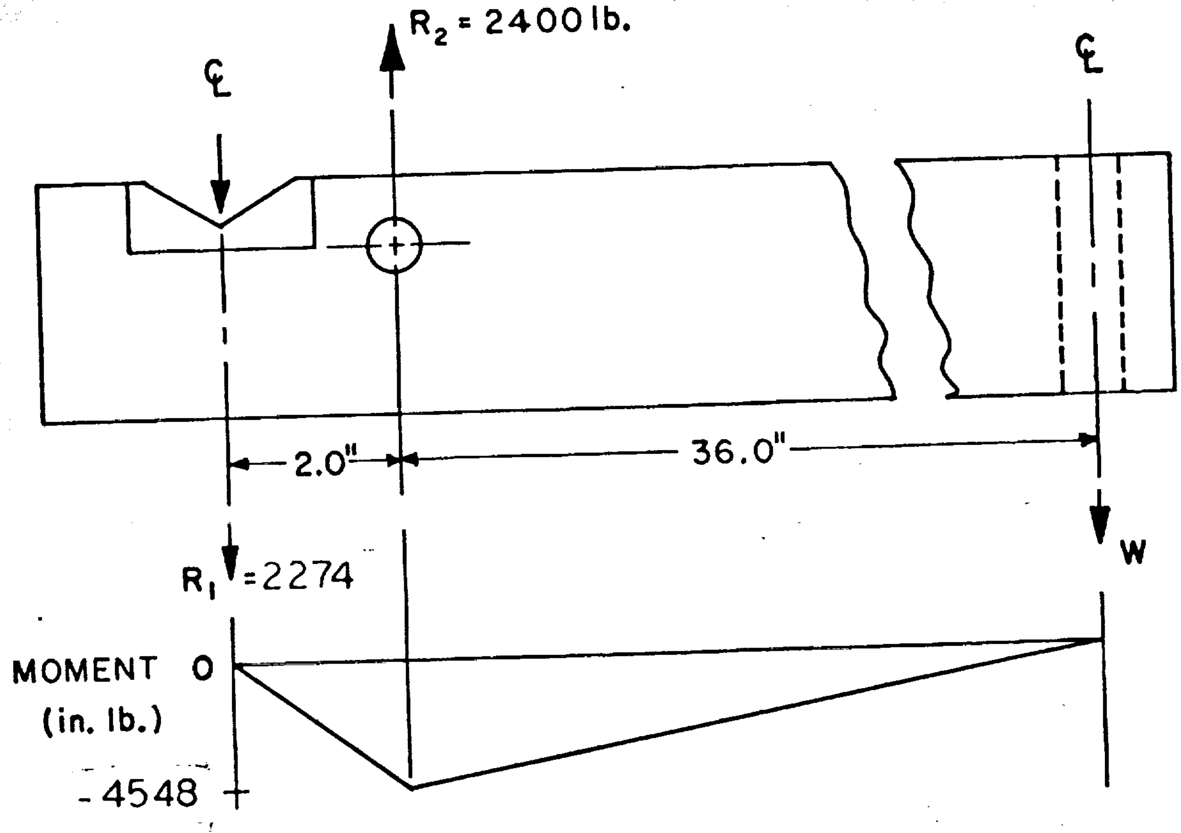
From Figure 6.12-b,

$$I = \frac{1}{12}(b)(h_o^3 - h_i^3) - Ad^2 \qquad (6.4.17)$$

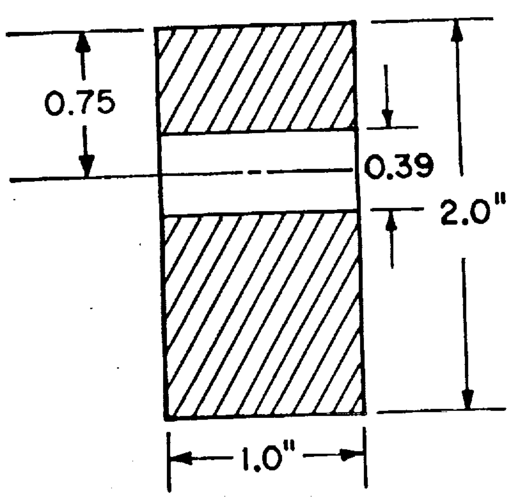
$$I = \frac{1}{12}(1)\left[2^3 - (3/2)^3\right] - (1.0)(.39)(.31)^2$$

$$I = 0.745 - .0376$$

$$I = 0.708 \text{ in}^4$$



(a) BENDING MOMENT DIAGRAM



(b) SECTION AT MAXIMUM MOMENT

FIGURE 6.12 LOADING LINK DIAGRAM

Then,

$$\sigma = \frac{Mc}{I} \qquad (6.4.16)$$

$$= \frac{(4548)\ (1.06)}{.708}$$

$$\sigma = \boxed{6750 \text{ psi}}$$

$$\text{Factor of Safety} = \frac{62}{6.75} = 9.2$$

Fulcrum unit and block: The fulcrum unit, D/N UANASA-6700-B-012, was the knife edge used as a pivot point for the lever arm. This unit mounts on the accessory plate and can be moved. The fulcrum block, D/N UANASA-6700-A-017, was the receptacle for the fulcrum point.

Electrical system. The electrical system provides for the test machine 440 V, 3 phase, 60 cycle power to the switchbox. From the switchbox the power goes to the motor. An automatic shutdown circuit is utilized, D/N UANASA-6700-D-003, to release power to the motor when the test specimen breaks. This circuit taps one of the 110 V lines of the switchbox for its controlling power.

Indexer Flange. A mating unit had to be designed to couple the Infinit-Indexer to the Falk gear reduction units on the back shaft. The mating unit was called an indexer flange, D/N UANASA-6700-C-007 and Figure 6.13.

Calculations

The bending loads are negligible.

Shear stress in hub with keyway:

$$T_{LSS} = T_{HSS} \times \text{Gear Ratio}$$

$$= 5400 \times 1.84$$

$$T_{LSS} = 9950 \text{ in-lb}$$

$$\tau = \frac{Tc}{J} \qquad (6.4.4)$$

$$J = \frac{\pi d^4}{32} \qquad (6.4.22)$$

But,

$$J = \frac{\pi}{32} (d_o^4 - d_i^4)$$

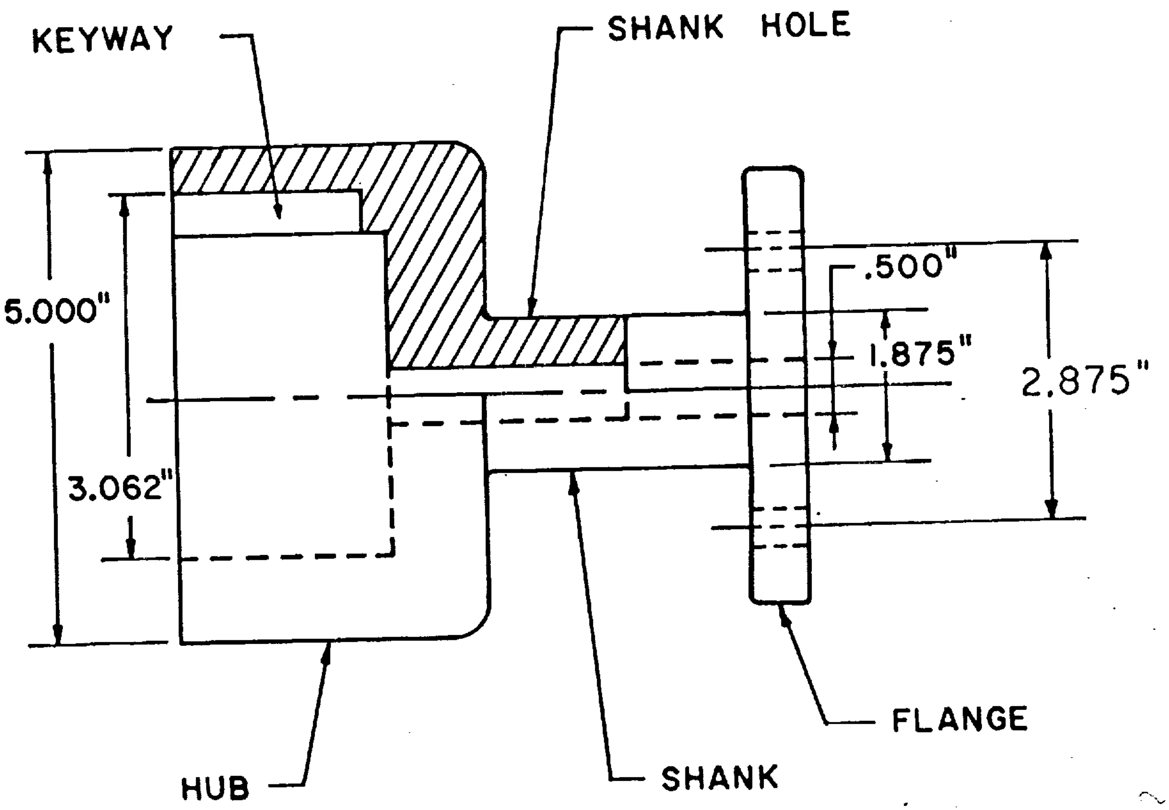


FIGURE 6.13 INDEXER FLANGE

where:

$$d_o = 5 \text{ in.}$$

$$d_i = 3.062 \text{ in.}$$

$$J = \frac{\pi}{32} (5^4 - 3.062^4)$$

$$J = 52.7 \text{ in.}^4$$

Substituting,

$$\tau = \frac{(9950)(2.5)}{52.7}$$

$$\tau = 472 \text{ psi}$$

$$\text{Factor of Safety} = \frac{62}{0.472} = 131$$

Shear stress in shank with hole:

$$\tau = \frac{Tc}{J} \qquad (6.4.4)$$

$$J = \frac{\pi}{32} (1.875^4 - 0.5^4) \qquad (6.4.22)$$

$$J = 1.21 \text{ in.}^4$$

$$\tau = \frac{(9950)(0.937)}{1.21}$$

$$\tau = 7700 \text{ psi}$$

$$\text{Factor of Safety} = \frac{62}{7.7} = 8.6$$

Shear stress in bolts in flange:

There are 6-3/8 - 24NF bolts used when coupling the flange to the Infinit-Indexer.

$$6 F R = T \qquad (6.4.23)$$

$$F = \frac{(9950)}{(6)(1.44)}$$

$$F = 1150 \text{ lb}$$

$$\tau = \frac{F}{A}$$

$$A = 0.081 \text{ in.}^2$$

$$\tau = \frac{1150}{0.0809}$$

$$\tau = 14\ 200 \text{ psi}$$

With SAE Grade 2 bolt the proof load = 55 000 psi (62, p. 247).

$$\text{Factor of Safety} = \frac{55}{14.2} = 3.88$$

Back shaft: The back shaft is the low speed shaft and only carries torque. The ASME code for shaft analysis was adhered to.

Calculations

$$\tau_d = \frac{16}{\pi d_s^3} \left[C_m M^2 + C_t T^2 \right]^{\frac{1}{2}} \quad (62, \text{ p. } 478) \qquad (6.4.24)$$

where:

T = 9950 in-lb. $\cong$ 10 000 in-lb.

C_t = 1.0 (62 p. 478) rotating shaft - steady torque

d_s = 2 in.

The bending moment was negligible, therefore consider $C_m M^2 = 0$.

$$\tau_d = \frac{16}{\pi (2)^3} \left[(1.0)(10\ 000)^2 \right]^{\frac{1}{2}}$$

$$\tau_d = 6370 \text{ psi}$$

τ_d should not be greater than the smaller of:

$$\tau_{d_{ALL}} = 0.30\ S_y$$

or

$$\tau_{d_{ALL}} = 0.18\ S_{UT} \quad (62, \text{ p. } 477)$$

Using SAE C1020, cold drawn steel with,

$$S_y = 79\ 000 \text{ psi}$$

$$S_{UT} = 80\ 000 \text{ psi}$$

$$\tau_{d_{ALL}} = (0.18)(80,000)$$

$$\tau_{d_{ALL}} = 14\,400 \text{ psi}$$

$$\text{Factor of Safety} = \frac{14,400}{6.37} = 2.26$$

Therefore any material better than SAE C1020 is satisfactory.

Accessory Components. - The following components were manufactured but do not require structural analysis.

1. Slip ring bushing
2. Slip ring brushes mount
3. Mechanical counter mount
4. Counter bushing

CHAPTER 6.5

INSTRUMENTATION

General Equipment

Strain gages.-Strain gages will be used to measure the strains in the specimen induced by the applied torque and bending moment. The strain gages selected are of three dynamic types, one for torque, and two for bending moment.

It is very difficult to place strain gages within the confines of the specimen groove for all test specimens. Therefore a correlation will be developed between the groove of the test specimen and the shank of the specimen holder for the bending moment data, thereby not requiring strain gages in all specimen grooves. The selected strain gages are presented in Table 6.5.

Slip rings.-Slipsrings are to be used as a means of bringing the strain signal from the rotating machinery to carrier amplifiers. The selected slip rings and their brushes are Breeze AJ-8005-A8. The manufacturer suggested this slip ring since it has a maximum number of contacts per ring and low electrical noise. Figure 6.14 shows the strain gages, for torque and bending moment, and the slip ring locations on the test machine.

Amplifier-Galvanometer-Recorder unit.-When the amplifier-galvanometer-recorder unit was selected, selection started with the amplifier. By selecting the Honeywell Model 119 carrier amplifier system to amplify the strain signal a companion set of galvanometers were recommended by the manufacturer, namely the M1650 galvanometer. Also when the galvanometers were selected the manufacturer recommended the Model 906 C-1 recorder. The advantage of this type of selection of equipment was that each unit was matched to the other. Table 6.5 presents the selected equipment.

Calculations for Bridge Output

Bridge output for torque.-A conventional arrangement for dynamic torque measurements was obtained from Perry and Lissner (63, p. 208). This arrangement, shown in Figure 6.15, compensates for temperature, eliminates effects of axial and bending strains, and minimizes inaccuracies due to contact resistance variations.

Calculations

The strains in the adjacent legs are subtracted algebraically and in the opposite legs are added algebraically. For an applied torque, the strains

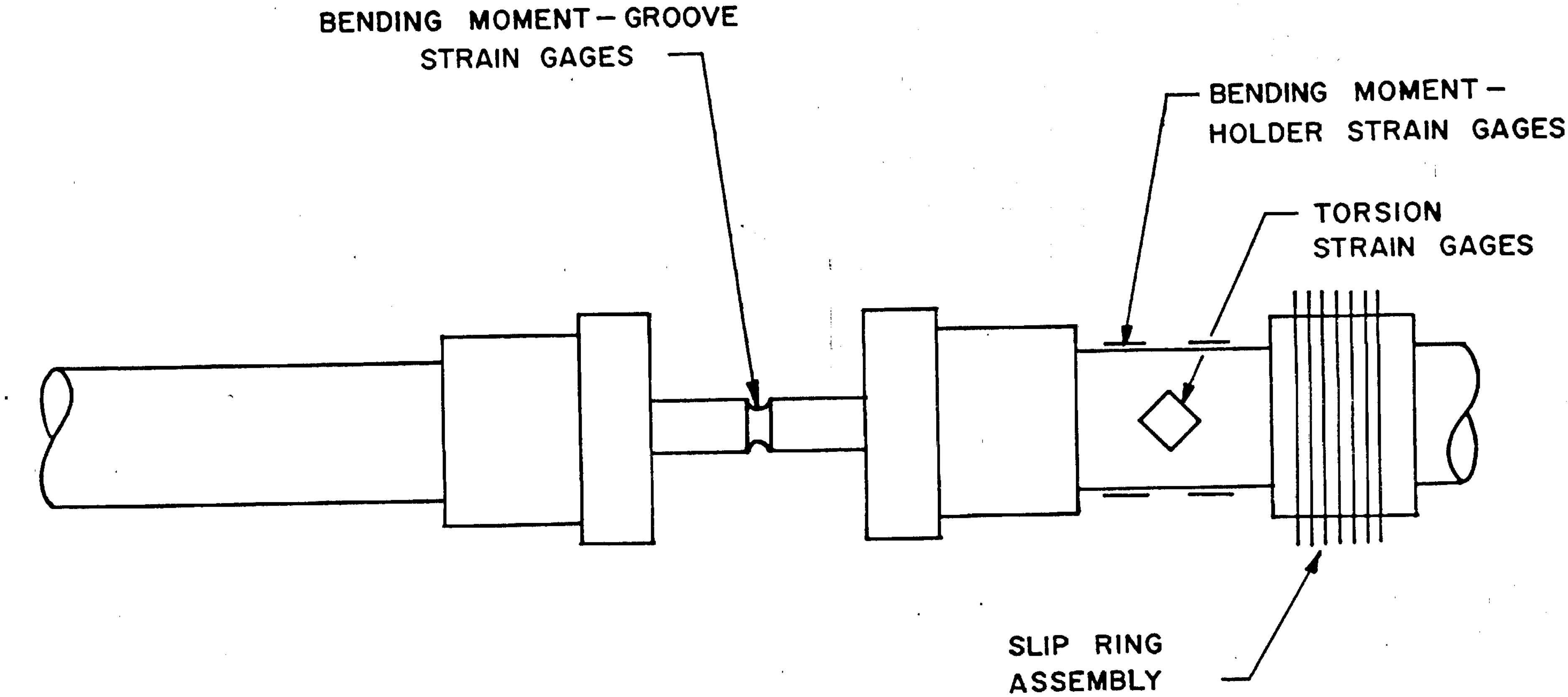
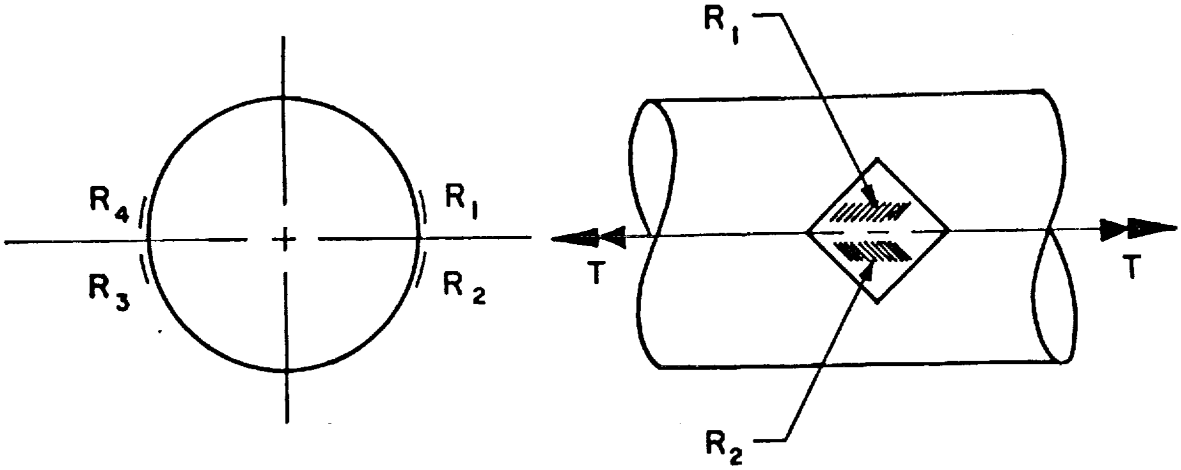
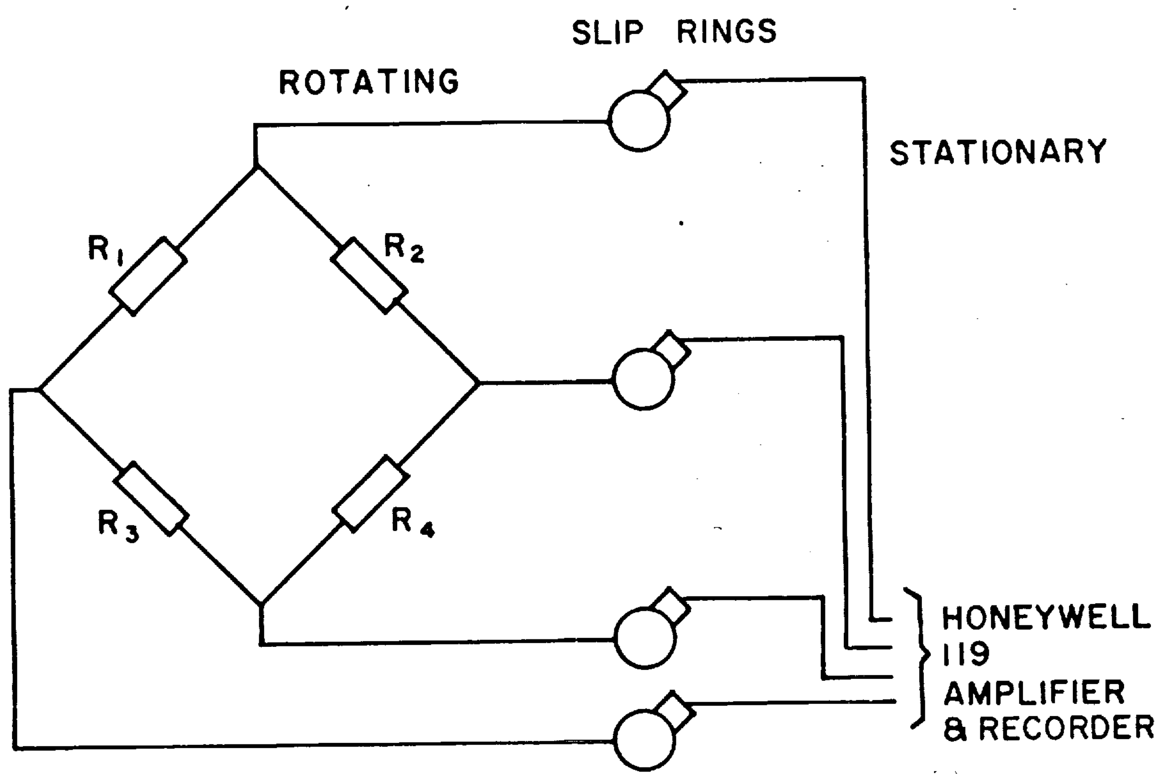


FIGURE 6.14 STRAIN GAGE AND SLIP RING ARRANGEMENT



(a) TORSION STRAIN GAGE



(b) STRAIN GAGE BRIDGE ARRANGEMENT

FIGURE 6.15 TORQUE INSTRUMENTATION

TABLE 6.5

INSTRUMENTATION

Name	Part Number	Manufacturer	Catalog Number	Qty.	Remarks
Metalfilm strain gages with leads	324B-190	The Budd Co. Instruments Division	BG 2400	4	Bending moment-shank of holder
Metalfilm strain gages with leads	3 x 4 - M15E-240	"	"	2	Bending moment groove of specimen
Metalfilm strain gages with leads	C6-121-R2VC	"	"	2	Torque
Slip rings and brushes	AJ-8005-AS	Breeze Corporation, Inc.	66SR	1	Transfer of data
Viscorder	906 C-1	Honeywell	D-2009	1	With grid line system, 14 magnetic assembly channel 0-5000 cps
Galvanometer	M1650	"	D-2007	1	
Amplifier	119	"	D-2005	1	Carrier and linear/integrating system with carrier channels 0-5000 cps

will be in proportion to the change in resistance of each gage or

$$\Delta R_1 \doteq +\epsilon \qquad (6.5.1)$$

$$\Delta R_2 \doteq -\epsilon \qquad (6.5.2)$$

$$\Delta R_3 \doteq +\epsilon \qquad (6.5.3)$$

$$\Delta R_4 \doteq -\epsilon \qquad (6.5.4)$$

Then the bridge output (BO) will be proportional to:

$$BO \doteq \Delta R_1 - \Delta R_2 + \Delta R_3 - \Delta R_4$$

$$\doteq \epsilon - (-\epsilon) + \epsilon - (-\epsilon)$$

$$BO_T \doteq 4\epsilon \qquad (6.5.5)$$

Equation (6.5.5) indicates that the bridge ouput will be four times the output of one strain gage.

For axial strains, the strains will be porportional to:

$$\Delta R_i \doteq +\epsilon \qquad i = 1, 2, 3, 4. \qquad (6.5.6)$$

Thus, axial strains will cancel out.

An estimation of the strain which a gage will measure was accomplished by considering the stresses that the toolholder would be subjected to, see Figure 6.16. The maximum strain in 2-dimensional stress is

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\mu \cdot \sigma_2}{E} \qquad (6.5.7)$$

but, in this case

$$\sigma_2 = -\sigma_1$$

then,

$$\epsilon_1 = \frac{\sigma_1 (1 + \mu)}{E}$$

However in pure shear

$$\sigma_1 = \tau \text{ numerically} \qquad (6.5.8)$$

Substituting,

$$\epsilon_1 = \frac{\tau (1 + \mu)}{E} \qquad (6.5.9)$$

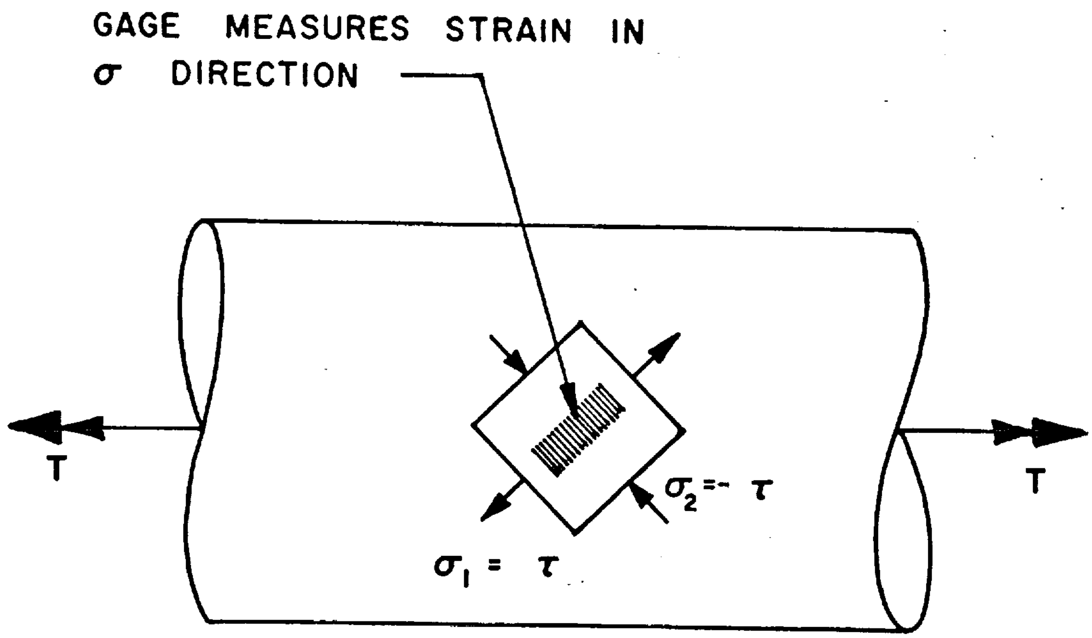


FIGURE 6.16 TORSIONAL STRESSES FOR ONE GAGE

The shear stress of the specimen holder was calculated, since the torque strain gage is mounted on the holder.

$$\tau = \frac{Tc}{J} \qquad (6.4.6)$$

From the calculations of the design torque to fracture a specimen,

$$T = 5400 \text{ in-lb}$$
$$c = 1 \text{ in.}$$

for a cylinder,

$$J = \frac{\pi}{32}(d_o^4 - d_i^4) \qquad (6.4.22)$$

$$= \frac{\pi}{32}(2^4 - 1.3125^4)$$

$$J = 1.28 \text{ in.}^4 \qquad (6.5.10)$$

$$\tau = \frac{(5400)(1)}{1.23} \qquad (6.4.6)$$

$$\tau = 4200 \text{ psi}$$

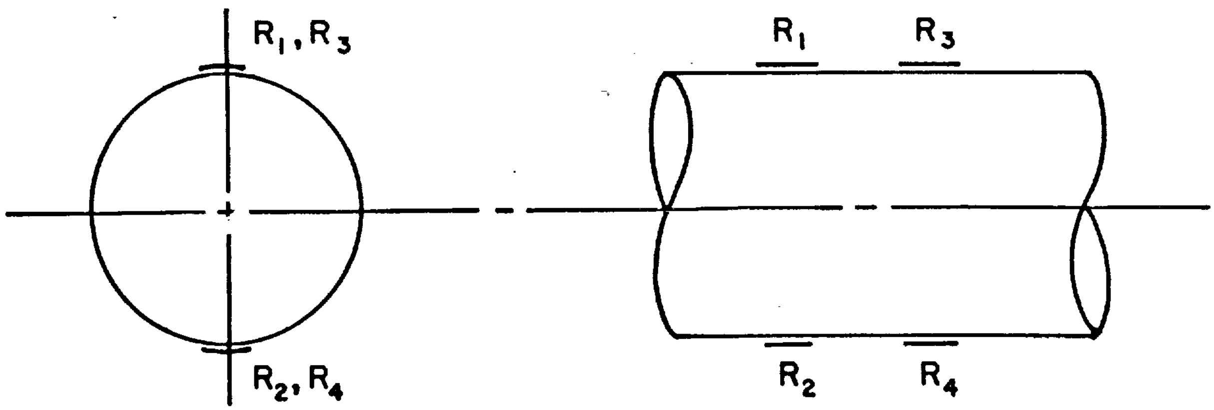
Substitute into Equation (6.5.9) for steel:

$$E = 30 \times 10^6 \text{ psi}$$
$$\mu = 0.27$$
$$\epsilon_1 = \frac{(4200)(1.27) \times 10^{-6}}{30}$$
$$\epsilon_1 = 173 \text{ microin/in.}$$

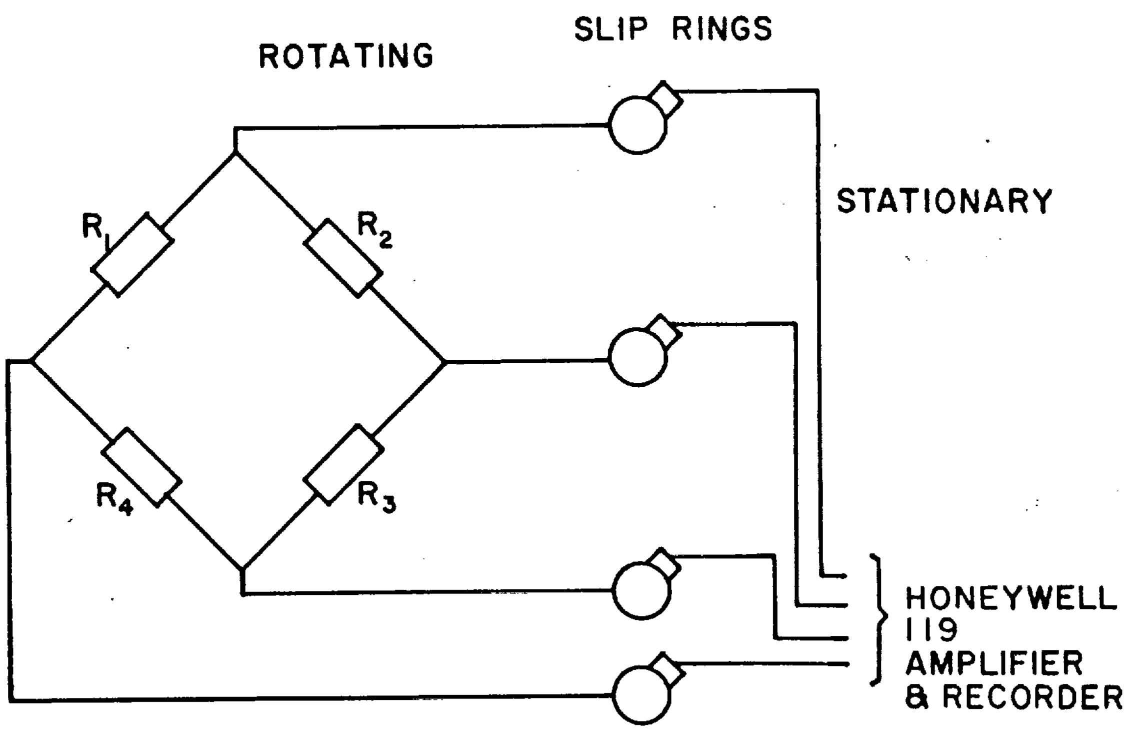
With four gages in the bridge circuit, the bridge output due to the torque would be

$$BO_T \doteq 4\epsilon$$
$$\doteq 4(173)$$
$$BO_T \doteq 692 \text{ microin/in.}$$

<u>Bridge output for bending moment.</u>-To get the desirable output from the strain gages in response to a pure bending moment, they must be mounted and electrically connected as shown in Fig. 6.17. Then for an applied torque, all of the strain gages, Fig. 6.17, for bending moment will sense a strain of the same sign and because $\epsilon_1 - \epsilon_1 + \epsilon_1 - \epsilon_1 = 0$, the torsional strain outputs will cancel as they should. A similar argument holds for axial strain outputs and they will cancel. The strain from the pure bending moment may be calculated as follows:



(a) BENDING MOMENT STRAIN GAGES



(b) STRAIN GAGE BRIDGE ARRANGEMENT

FIGURE 6.17 BENDING MOMENT INSTRUMENTATION

Calculations

$$\Delta R_1 \doteq +\epsilon \tag{6.5.11}$$

$$\Delta R_2 \doteq -\epsilon \tag{6.5.12}$$

$$\Delta R_3 \doteq +\epsilon \tag{6.5.13}$$

$$\Delta R_4 \doteq -\epsilon \tag{6.5.14}$$

$$BO_{BM} \doteq \epsilon -(-\epsilon) + \epsilon-(-\epsilon) \tag{6.5.15}$$

$$BO_{BM} \doteq 4\ \epsilon$$

The strain which a gage will measure can be estimated by

$$\epsilon = \frac{\sigma}{E} \tag{6.5.17}$$

where,

$$\sigma = \frac{Mc}{I} \tag{6.4.16}$$

From the design calculations for the fracture of a specimen in bending, the required moment is

$$M = 3540 \text{ in-lb}$$

$$I = \frac{\pi}{64} (d_o^4 - d_i^4) \tag{6.5.18}$$

and

$$I = \frac{J}{2} \tag{6.5.19}$$

Substituting the answer, Equation (6.5.10), into Equation (6.5.19),

$$I = \frac{1.28}{2} = 0.64 \text{ in.}^4$$

with

$$c = 1 \text{ in.}$$

$$\sigma = \frac{(3540)(1)}{0.64}$$

$$\sigma = 5550 \text{ psi} \tag{6.5.20}$$

Substituting Equation (6.5.20) into Equation (6.5.17),

$$\epsilon = \frac{5550}{30 \times 10^6}$$

$$\epsilon = 184 \text{ microin/in.}$$

then the bridge output due to the bending moment would be

$$BO_{BM} \doteq 4\epsilon$$

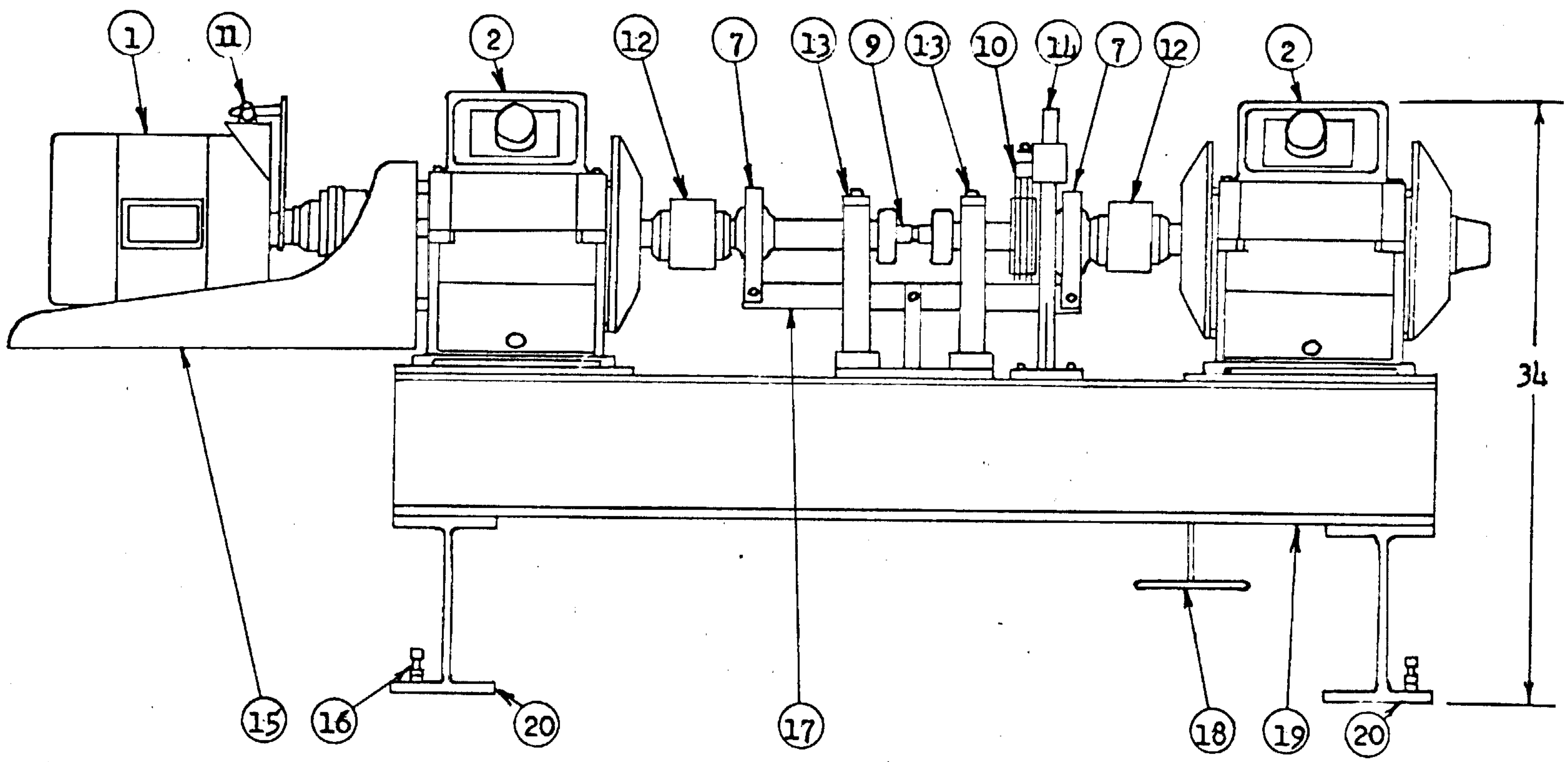
$$\doteq 4(184)$$

$$BO_{BM} \doteq 736 \text{ micro in/in}$$

The maximum sensitivity of the amplifier can produce a full scale deflection with a strain of 50 micro inches. The torque measurements will utilize the recording paper with the minimum at one side and the maximum at the other side. The sensitivity ratio is the measured strain to the actual rated strain of the instrument for full scale deflection. Then for maximum torque strain the sensitivity ratio is 13.8. Similarly, for maximum bending moment strains the sensitivity ratio is 29.4. These are highly acceptable values because they will provide high accuracy of measurement of both the bending and torsional stresses involved.

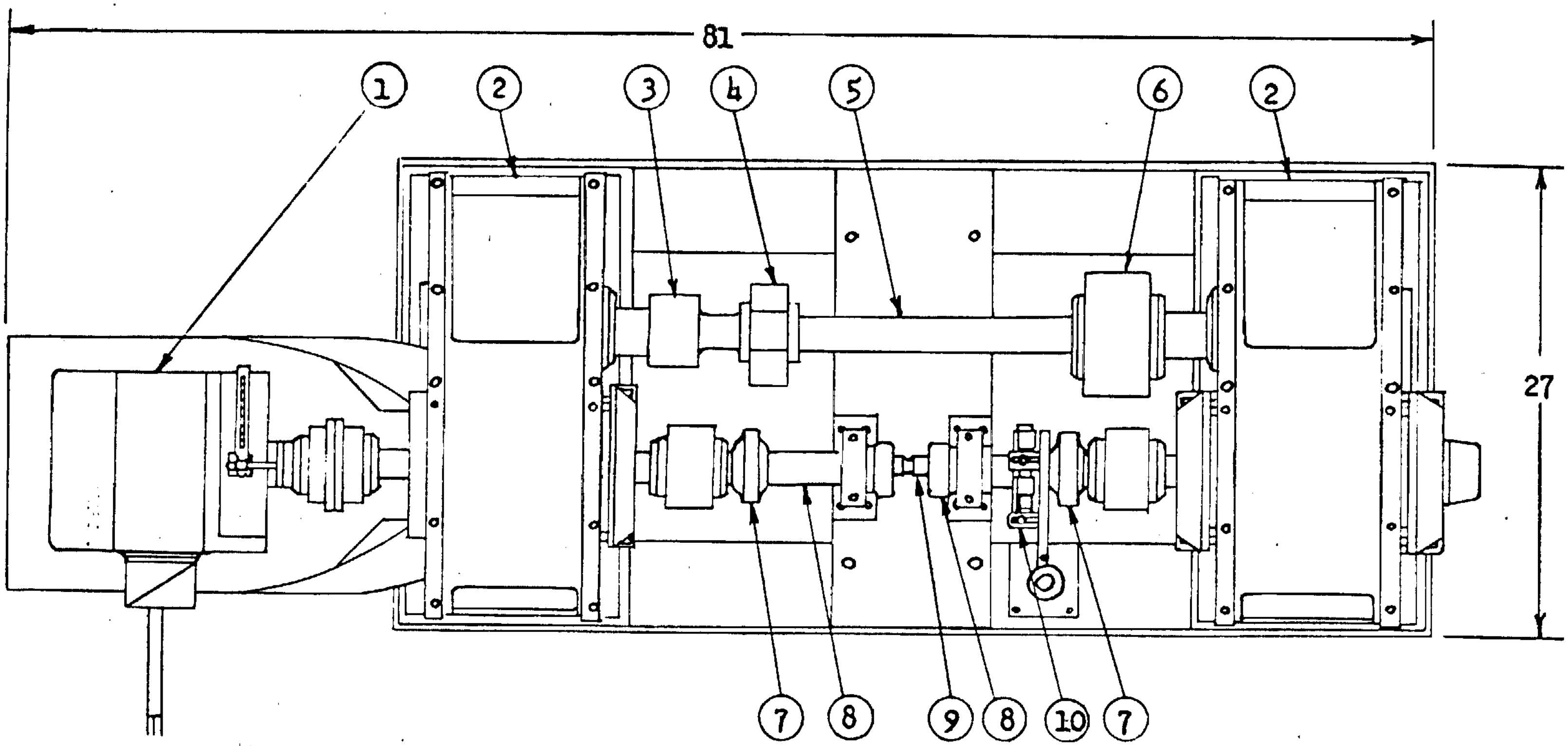
Machine Drawings

Figures 6.18 and 6.19 show the overall machine. Figure 6.18 shows the front view of the complex-fatigue reliability research machine and Figure 6.19 shows the top view. The major components involved are identified as well as the width, height and length of these machines.



1. Motor
2. Gearboxes
7. Loading Frame Bearing
9. Test Specimen
10. Slip Rings and Brushes
11. Counter
12. Coupling
13. Safety Bridge
14. Brush Holder
15. Motor Bracket
16. Mount Bolts
17. Loading Frame
18. Weights
19. Machine Mount
20. Floor Mounts

FIGURE 6.18 COMPLEX-FATIGUE RELIABILITY RESEARCH MACHINE - FRONT VIEW



1. Motor
2. Gearboxes
3. Coupling
4. Infinite Indexer
5. Backshaft
6. Coupling
7. Loading Frame Bearing
8. Straight Shank Toolholder
9. Test Specimen
10. Slip Rings and Brushes

FIGURE 6.19 COMPLEX-FATIGUE RELIABILITY RESEARCH MACHINE - TOP VIEW

CHAPTER 6.6

COST ANALYSIS

The total cost of the fatigue test machine, as given in Table 6.6, is $3572.95. This figure does not include the cost for the test specimens and the instrumentation external to the machine, presented in Chapter 6.5. The University of Arizona fabrication cost figure is for the material and labor of those components that needed to be manufactured. The cost of one specimen of the type specified in Table 6.1 is $3.05. The total cost of the instrumentation, external to the machine, as given in Table 6.7, is $4728.00.

TABLE 6.6

COST ESTIMATE FOR TEST MACHINE
BASED ON ONE MACHINE

ITEM	SUPPLIER AND PART NO.	NUMBER REQUIRED	COST PER ITEM	TOTAL COST
Tool holder	Balas S16-8"-C-12	2	$75.00	$150.00
Collets	Balas C-12: 3/4"	2	16.00	32.00
Coupling, front shaft	Sier-Bath No. 7	2	40.00	40.00
Bearings, loading	SKF 22211-CK	2	21.00	42.00
Bearing adapter	SNW-11 x 2	2	3.00	6.00
Gear reducer	a) Falk 2050 Y1: 1 Assy. No. 20 1 Assy. No. 12 b) Cooling fans with each unit c) Drilled motor bracket d) Falk 6F coupling	1	1891.00	1891.00
Coupling, back shaft	Sier-Bath No. 3	1	94.00	94.00
Infinit-Indexer	United Shoe Machinery HD UI	1	180.00	180.00
Strain gages	Budd 324 B-190	1 pkg.	37.00	37.00
	C6-121R2VC	1 pkg.	75.00	75.00
	3X4M15E-240	1 pkg.	100.00	100.00
Slip rings and brushes	Breeze AJ-3005-A8	1	131.00	131.00

TABLE 6.6 - Continued

COST ESTIMATE FOR TEST MACHINE
BASED ON ONE MACHINE

ITEM	SUPPLIER AND PART NO.	NUMBER REQUIRED	COST PER ITEM	TOTAL COST
I-Beam	8SWF 17 x 60	2	$ 13.00	$ 26.00
3/8 in. Plate	26" x 36"	1	16.00	16.00
Electric Motor[a]	G.E. 5K213AK 202	1	175.00	175.00
Motor Starter[a]	G.E. CR 108C 104 ABA	1		
Counter Revolution	Veeder Root Series 111518	1	45.65	54.65
Limit Switch	Honeywell Micro Switch BZ [illegible]	1	3.00	3.00
The University of Arizona	Fabrication	-	-	300.00
The University of Arizona	Electrical Shop	-	-	90.00
Positive Drive	Durkee-Atwood			8.00
Pulley	30XL037	1		
Pulley	60XL037	1		
Belt	220XL037	2		
Paint	Rust-oleum	-	-	5.00
Miscellaneous Bolts, Nuts	-	-	-	40.00
Lubrication	-	-	-	20.00
Special Tools Spanner wrench Grease gun Grease fittings	-	-	-	16.00
TOTAL				$3572.95

[a]Donated by General Electric Co., Phoenix, Arizona

TABLE 6.7

COST OF INSTRUMENTATION

ITEM	SUPPLIER	PART NO.	NUMBER REQUIRED	COST	
Basic Recording Oscillograph	Honeywell	906C-1650	1	$1325.00	
Grid Line System	Honeywell	901343	1	200.00	
Miniature 14-channel Magnet assembly	Honeywell	-	1	350.00	
Subminiature Galvanometer	Honeywell	M1650	2	270.00	×4 = 1,080
Carrier Amplifier, Basic system less channels	Honeywell	Model 119	1	1600.00	
Carrier channels	Honeywell	Model 119B-1	2	800.00	×4 = 3,200
Kodak Linagraph Direct print paper	Eastman-Kodak	100338	20 bxs	183.00	
TOTAL				$4728.00	
				4280	
				$9,008	

Total Equipment cost: $20,000
Three machines & recorder system

CHAPTER 6.7

CONCLUSIONS

The major effort as reported in this section was to design, develop, and build a combined-stress fatigue machine at The University of Arizona. The combined stresses being combined are those from reversed bending and steady torque.

The initial design of the fatigue machine utilized a 1:1 gear ratio. The gears and required equipment would have had to be special order items. The required gear box was too large and the associated lubrication equipment quite complicated. Therefore commercially available gear reduction units were selected for this research application.

The initial design speed of the fatigue machine was 3600 rpm. Major problems arose because of this high rpm. The gear reduction units became extremely large and cumbersome, thus a large induction type motor (40 hp) would have been required. With these facts in mind the rpm was reduced to 1800.

The four-square principle is used to apply the torque. The major advantage of this principle is that the motor supplies only the losses present within the gear reduction units.

The fatigue machine is capable of applying 150 000 psi of bending and 110 000 psi of torsional stress on a one-half inch nominal diameter, SAE 4340, cold drawn heat treated to Rockwell "C" 35/40 steel test specimen. The combined-stress fatigue machine was designed for 5400 in-lb of torque and 3540 in-lb of bending moment.

The specimen is held by a set of collets. These collets are not capable of holding the test specimen from slipping when the maximum amount of torque is applied to the system. Therefore a keyway has been machined into each end of the test specimen.

The major portion of this fatigue machine consists of commercially available, time tested components. Therefore the machine could be readily reproduced, and the cost is relatively inexpensive. The total cost of the Combined Bending and Torsion Fatigue Machine for Reliability Research is less than $3600.00.

Two of these machines have already been built and instrumented and are presently operating satisfactorily, except for some problems encountered in the mounting and reliable operation of the strain gages.

CHAPTER 6.8

RECOMMENDATIONS

The fatigue machine designed for combined stresses closely simulates field conditions. However with certain modifications this fatigue machine could come even closer to certain field conditions. Considerations for redesign are as follows:

1. A method of shock loading the test specimen to simulate sudden loads some shaft might undergo.

2. A method of varying the torque and the bending moment, during the test operation. These conditions should be varied simultaneously if maintaining the same stress ratio is required.

3. Incorporate environments in which an actual shaft operates. For example, if the shaft is operating at high temperature, a means of heating the test specimen can be developed.

4. Locate a commercial collet that would hold steady torque in excess of 6000 in-lb, thereby allowing less machining on the test specimen than is required at present.

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SECTION 7

EXPERIMENTAL PROGRAM TO DETERMINE STRENGTH SURFACES FOR DESIGN BY RELIABILITY

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CHAPTER 7.1

INTRODUCTION

This project has a twofold objective. The first is to provide a theory and methodology for designing by reliability in complex fatigue; the second is to provide experimental determination of the required strength surfaces, and, concurrently, a verification of the practicality of the methodology.

The results which have been accomplished to date in theory and methodology have been presented in Sections 1 through 5. The experimental research equipment has been described in Section 6.

This section concentrates on describing the experimental research program, the results obtained to date, their analysis and their conversion to strength distributions. The direction of the second year's research effort is also presented.

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CHAPTER 7.2

CALIBRATION OF RESEARCH EQUIPMENT

Required Calibration

Toolholder strain gages.- In order to have confidence in the output of the recording equipment, calibration of the toolholder strain gages is required. This calibration is necessary for both the bending and torsion bridges. By comparing the recorded strain output with the output expected from theory, any significant errors can be spotted.

Specimen groove stress.- Also required for the testing program is a calibration between the stress level on the toolholder and the actual stress level in the specimen groove. Once this calibration has been accomplished, there will exist a curve for predicting specimen stress vs. toolholder strain. Then it will no longer be necessary to measure the actual stress in each specimen groove - a costly and time-consuming step.

The procedure used in calibration and the results to date are described below.

Calibration Procedure

Toolholder bending gages.- The first step in the calibration program was to compute the expected strain output from the toolholder due to the dead weight of the loading frame and due to weights being applied at the end of the loading link. Although the calculation involved in such a procedure will not lead to "exact" answers due to uncertainties about lever arms, moduli of elasticity, etc., they will serve to point out any significant errors in instrumentation or gaging.

Figure 7.1 shows schematically the arrangement of the loading mechanism. The weight of each item and the total weight are also shown. In Table 7.1 the expected strain from the loading frame and the weights on the end of the loading link have been computed.

To compare the actual output from the bending strain gage bridge with the output predicted in Table 7.1, tests were run on the machines and the strain output was recorded with the instrumentation described in Section 6. The recommendations of the Honeywell Company, as contained in their Instruction Manuals (1) and (2)*, were adhered to in setting up, operating, and calibrating the equipment.

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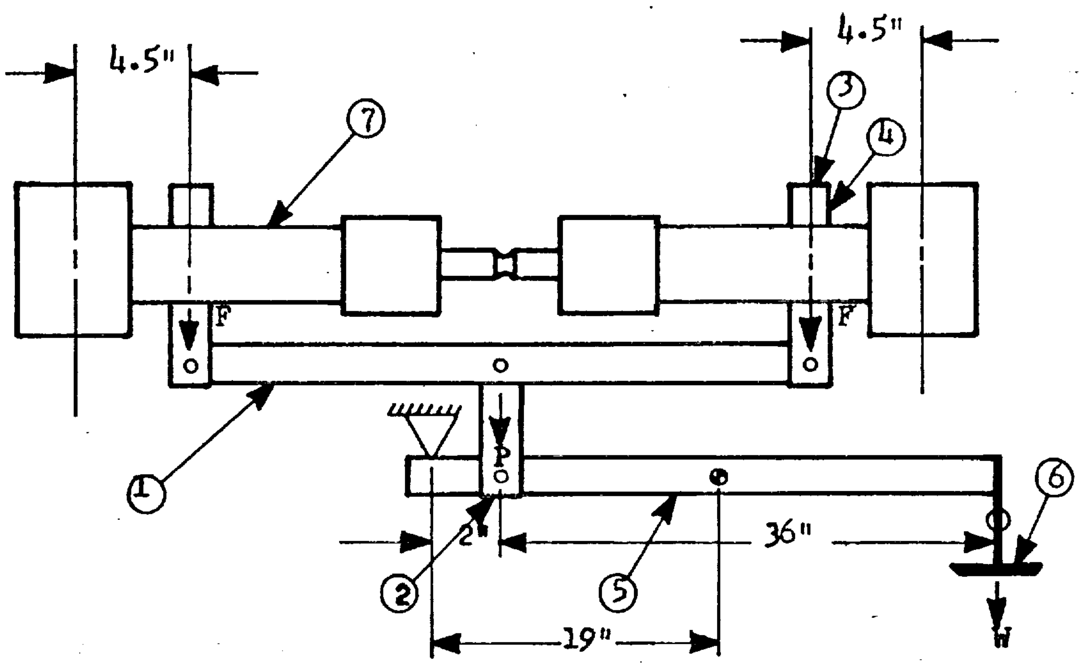
* Numbers in parenthesis refer to References at the end of this section.



Item	Weight of Items (lb)	No. of Items	Total Weight (lb)
1. Horiz. Link	7.8	1	7.8
2. Vert. Link	1.3	2	2.6
3. Br'g. Ring	4.2	2	8.5
4. Br'g. Ass'y.	2.6	2	5.2
5. Loading Bar	22.5	1	22.5
6. Loading Pan	1.5	1	1.5
7. Toolholder	9.0	2	18.0

FIGURE 7.1 MOMENT ON SPECIMEN CALCULATED FROM WEIGHT OF LOADING BAR, LOADING BEARINGS, WEIGHTS ON PAN, ETC.

TABLE 7.1

EXPECTED STRAIN ON TOOLHOLDER DUE TO DEAD WEIGHT OF LOADING FRAME AND WEIGHTS APPLIED THEREON

Moments Due To Frame Items

ITEM	WEIGHT (lb)	LEVERAGE FACTOR	LOAD P (lb)	LOAD F (lb)	TOOLHOLDER MOMENT (in-lb)	TOOLHOLDER STRESS (psi)	TOOLHOLDER STRAIN (μ in/in)
1. Horiz. Link	7.8	1.0	7.8	3.9	17.6	27.6	0.93
2. Vert. Links	2.6	1.0	2.6	1.3	5.9	9.2	0.31
3. Br'g. Ring	4.2	—	—	4.2	18.9	29.6	0.99
4. Br'g. Ass'y.	2.6	—	—	2.6	11.7	18.3	0.61
5. Load. Bar	22.5	9.5	213.0	106.5	479.0	749.0	25.00
6. Load. Pan	1.5	19.0	28.5	14.25	64.2	100.1	3.32
					597.3	933.8	31.16

Moments Due To Additional Weights On Loading Pan

ITEM	WEIGHT (lb)	LEVERAGE FACTOR	LOAD P (lb)	LOAD F (lb)	TOOLHOLDER MOMENT (in-lb)	TOOLHOLDER STRESS (psi)	TOOLHOLDER STRAIN (μ in/in)
1. 2.5 lb. wt.	2.5	19.0	47.5	23.75	107.0	167.0	5.57
2. 5.0 lb. wt.	5.0	19.0	95.0	47.5	214.0	334.0	11.10
3. 10 lb. wt.	10.0	19.0	190.0	95.0	426.0	665.0	21.80
4. 25 lb. wt.	25.0	19.0	475.0	237.5	1070.0	1670.0	55.57
5. 50 lb. wt.	50.0	19.0	950.0	475.0	2180.0	3340.0	111.00

Machine No. 1 was calibrated first in the following way: A recording of strain output was made with a specimen (and thus the toolholder) subjected to the dead weight of the loading frame alone. Next, a 2 1/2-pound weight was put on the loading pan, and another recording was made. Recordings were then made with increasing increments of 2 1/2 pounds of load up to twenty-five pounds, and then with decreasing increments of 2 1/2 pounds back down to the dead weight of the loading frame. The above process was repeated three times and the results were plotted as shown in Figure 7.2.

This curve shows that the measured strain agrees with that expected from theory very well, the maximum deviation being about 7 μ in/in. Also, the reproducability of the data is quite good. The data in most cases is within a band which is $\pm$ 2.5 μ in/in from the mean value. It was felt that the visicorder recordings could not be read any closer than this.

In Figure 7.3, the calibration curve of the mean values of Figure 7.2 is shown. Here the mean values are all well within the $\pm$ 2.5 μ in/in deviation, and the curve shows good linearity. This is the calibration curve for Machine No. 1, relating the load on the loading pan to the expected strain output in μ in/in from the bending bridge.

The above calibration scheme was repeated again for Machine No. 2. Figure 7.4 shows the final results, the mean of six values, plotted for Machine No. 2 and compared with Machine No. 1. Several conclusions can be drawn from this figure:

1. Both machines agree quite well with the output expected from theory.

2. The machines are very nearly identical. The average difference between them being 1.51 μ in/in.

3. Both machines show a linear relation between strain output and load on the loading pan, and the slopes of these lines for both machines are essentially identical.

During the testing program it was found that a shorter loading bar would be required in order to produce lower stress levels. The above program was repeated on both machines for the short loading bar. The results are shown in Figures 7.5, 7.6, and 7.7. The results and conclusions for these figures are much the same as those given above.

Calibration of stress in specimen groove.- Once the strain in the toolholder bending bridge is determined, the major concern becomes that of determining the specimen stress. Theoretically, expected values of specimen stress can be calculated from the known toolholder strain. The toolholder strain can be converted to toolholder stress, and the toolholder moment can be calculated from the toolholder stress. Since the moment on the toolholder and the moment on the specimen are the same, the

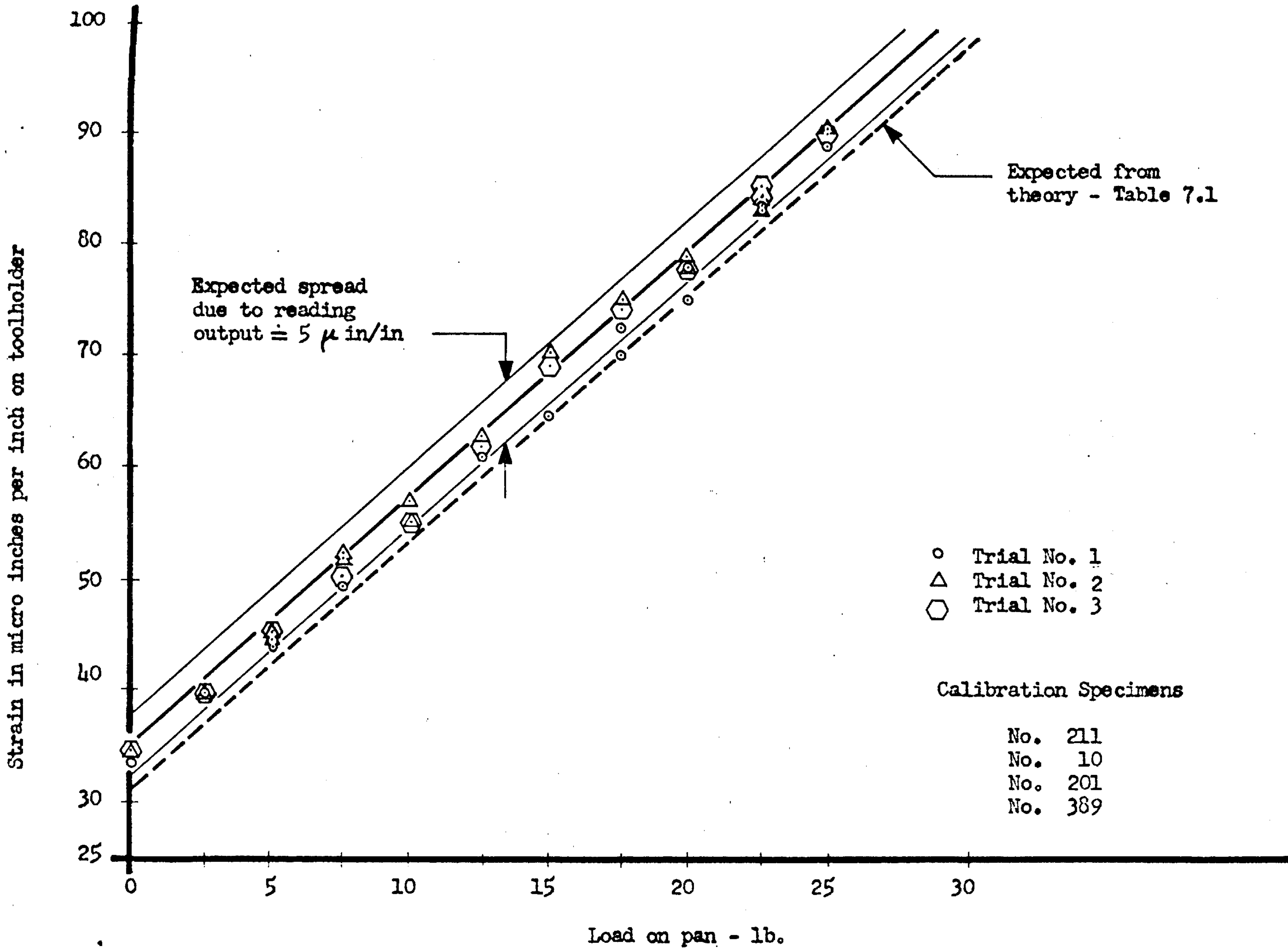


FIGURE 7.2 STRAIN RECORDED FROM BENDING BRIDGE VS LOAD ON LOADING PAN FOR MACHINE NO. 1 (LONG LEVER ARM)

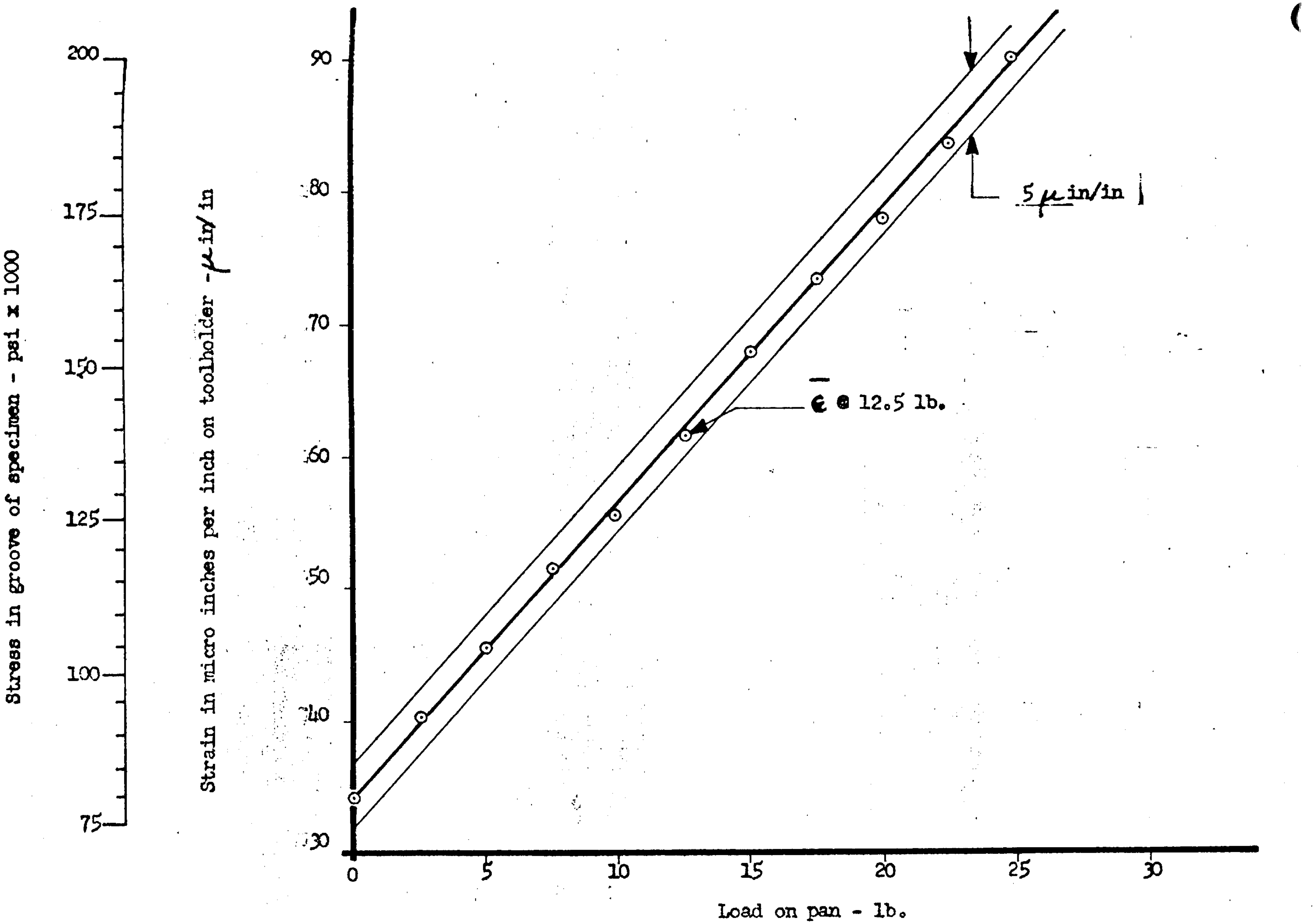


FIGURE 7.3 MEAN VALUES OF TEST DATA SHOWN IN FIGURE 7.2 PLOTTED VS LOAD ON PAN FOR MACHINE NO. 1
STRESS IN SPECIMEN GROOVE IS ALSO SHOWN ON ORDINATE
(LONG LEVER ARM)

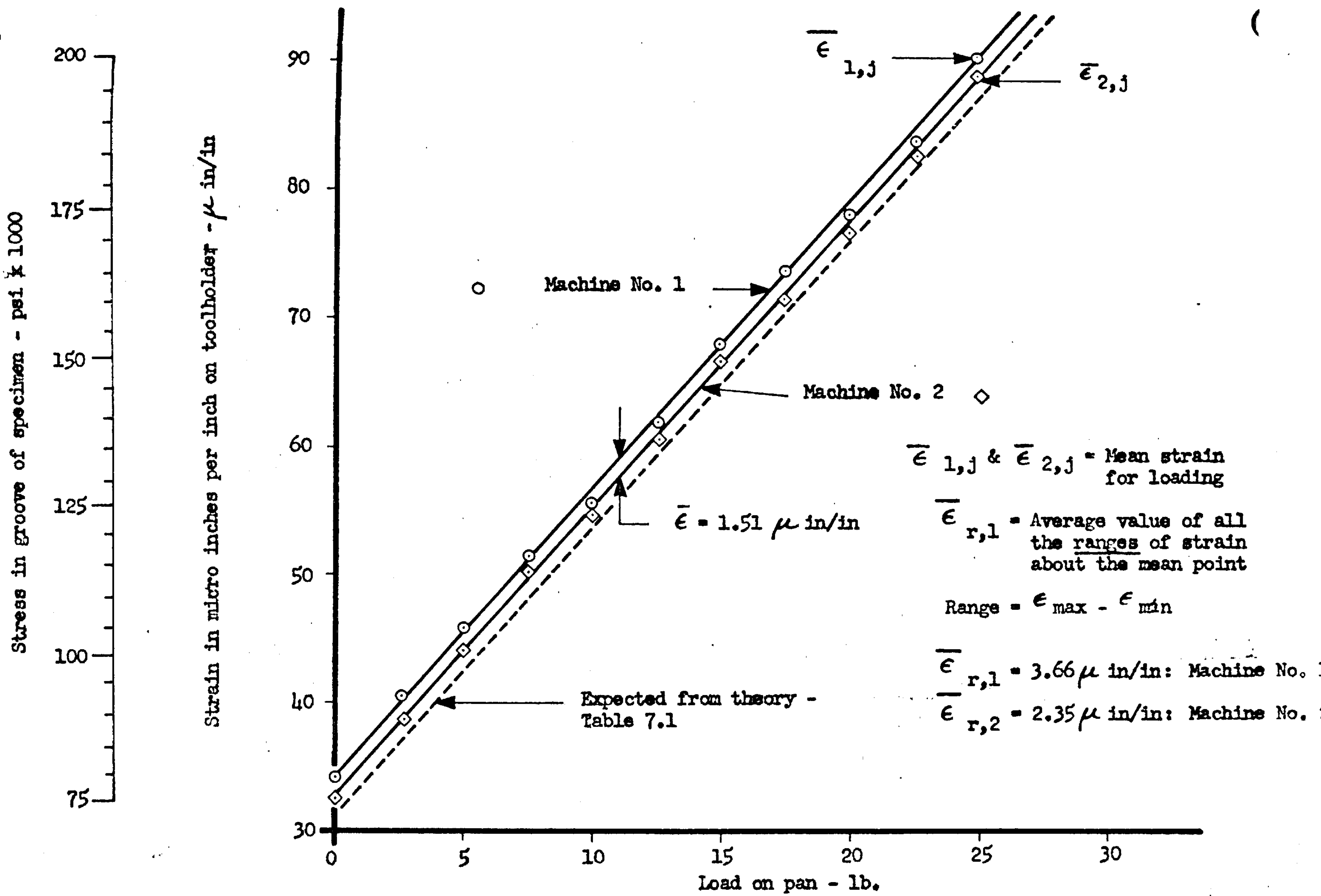


FIGURE 7.4 CALIBRATION CURVES FOR MACHINES 1 AND 2 BASED ON MEAN VALUE OF SIX OBSERVATIONS PER POINT. STRAIN ON TOOLHOLDER AND STRESS IN SPECIMEN GROOVE VS LOAD ON PAN ARE GIVEN. (LONG LEVER ARM)

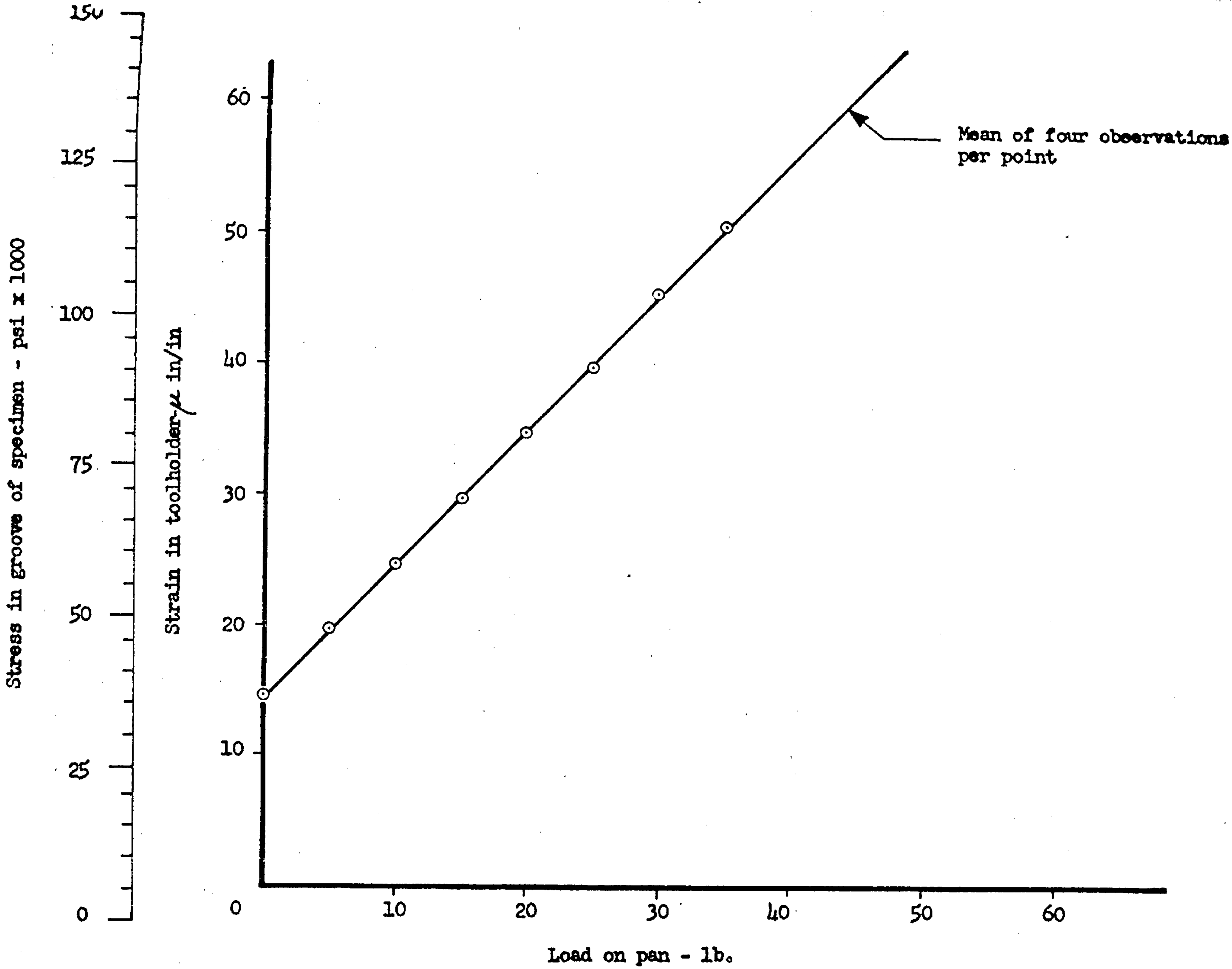


FIGURE 7.5 STRAIN IN TOOLHOLDER AND STRESS IN SPECIMEN GROOVE OF MACHINE NO. 1 VS LOAD ON PAN OF SHORT LEVER ARM

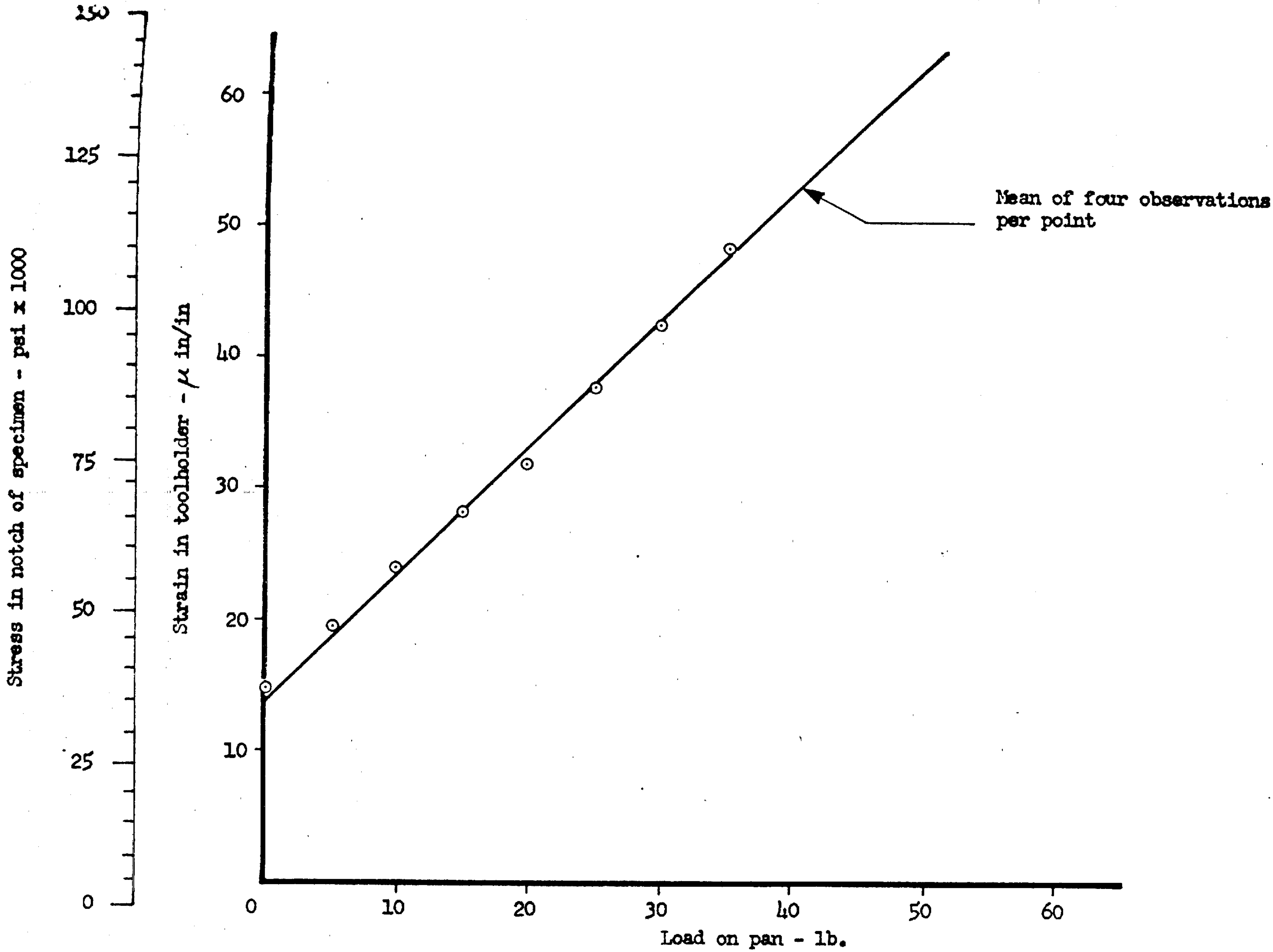


FIGURE 7.6 STRAIN IN TOOLHOLDER OF MACHINE NO. 2 VS LOAD ON PAN OF SHORT LEVER ARM

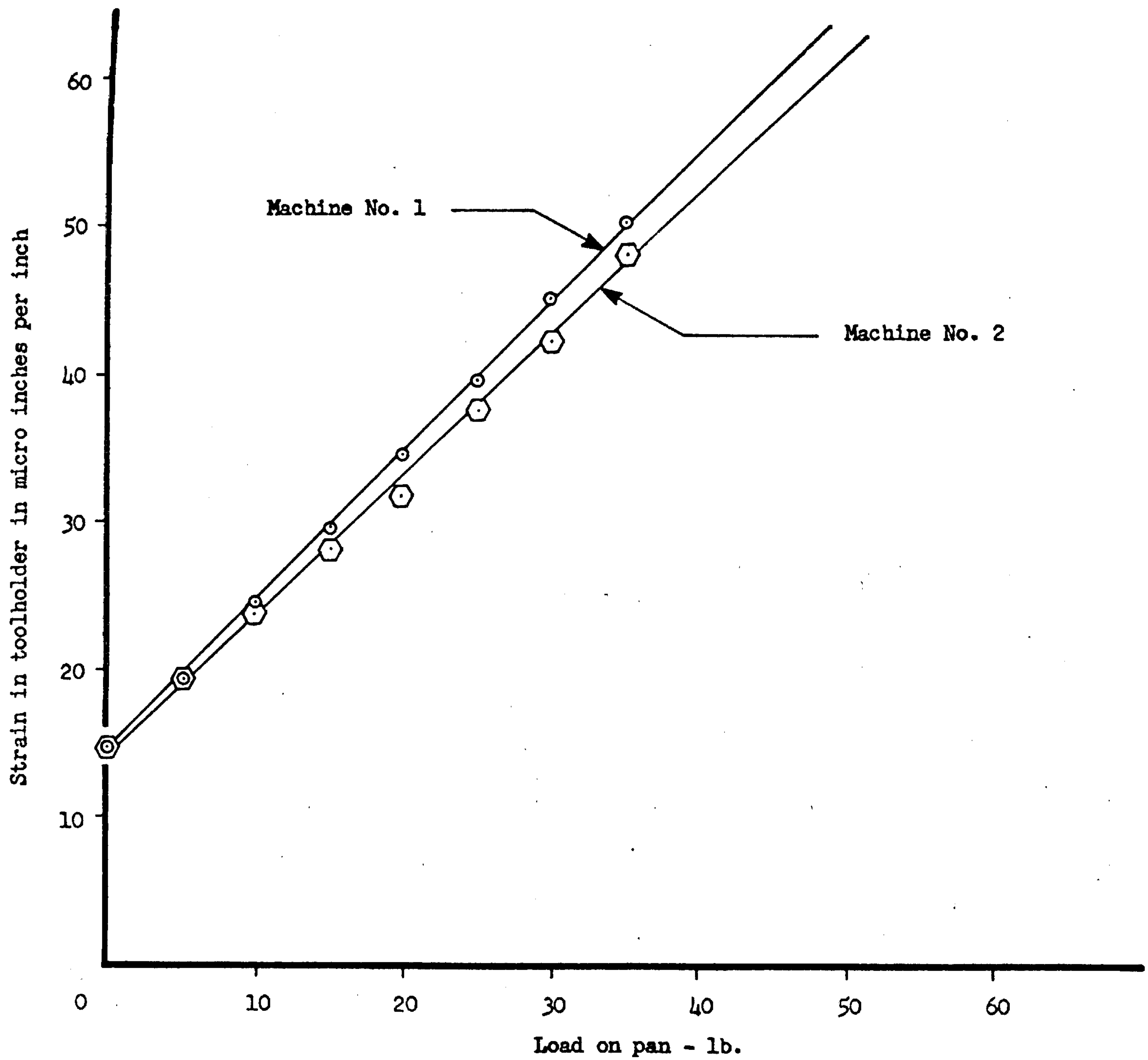


FIGURE 7.7 STRAIN IN TOOLHOLDER FOR MACHINES 1 AND 2 VS. LOAD ON PAN FOR SHORT LEVER ARM

specimen stress and strain can be calculated from theory as follows:

By a static analysis of Figure 7.1, the equation relating the weight on the loading pan to the moment on the specimen is

$$M_s = 207 + 21.4\ w \tag{7.2.1}$$

with the short lever arm, and

$$M_l = 597.3 + 42.7\ w \tag{7.2.2}$$

with the long lever arm, where

M = moment on the specimen in in-lb.

and

w = weight on the loading pan in lb.

By using the bending stress formula, and the geometry of the specimen as given in drawing No. UANASA-6700-B-002, the bending stresses in the specimen groove can be related to the moment, M, by

$$\sigma_{spec.} = \frac{K_t\ Mc}{I}$$

where

$$I/c = \frac{d^3}{32} = 0.0122\ in^3$$

d = 0.498 in (as per received specimens)

K_t = 1.41 (as per received specimens)

so that

$$\sigma_{spec.} = 115.5\ M \tag{7.2.3}$$

This equation gives the relationship between the stress in the specimen groove and the moment on the toolholder; which moment is equal to the moment on the specimen.

Combining Equations (7.2.1), (7.2.2) and (7.2.3) gives

$$\sigma_{spec.} = 115.5\ (207 + 21.4\ w)$$

$$= 23,800 + 2,460\ w \tag{7.2.4}$$

for the short lever arm, and

$$\sigma_{spec.} = 115.5\ (597.3 + 42.7\ w)$$

$$= 68,900 + 4,920\ w \tag{7.2.5}$$

for the long lever arm.

The specimen strain is related to the specimen stress by the formula

$$\epsilon_s = \frac{\sigma_s}{E_s} \tag{7.2.6}$$

The toolholder bending stresses can be related to the moment, M, on the specimen or toolholder by

$$\sigma_{tool} = \frac{Mc}{I}$$

$$= \frac{32 \times M \times 2}{\pi \left[(2)^4 - (1.3125)^4\right]}$$

which reduces to

$$\sigma_{tool} = 1.56\ M \tag{7.2.7}$$

Then the toolholder strain, which is the strain being monitored by the instrumentation, is given by

$$\epsilon_t = \frac{\sigma_t}{E_t} \tag{7.2.8}$$

By combining Equations (7.2.3) and (7.2.7) the relation

$$\sigma_{spec.} = \frac{115.5}{1.56}\ \sigma_{tool}$$

or

$$\sigma_{spec.} = 73.8\ \sigma_{tool} \tag{7.2.9}$$

is obtained.

Then by the use of Equations (7.2.6) and (7.2.8), it is seen that

$$\epsilon_{spec.} = 73.8 \, \epsilon_{tool} \frac{E_{tool}}{E_{spec.}}$$

Since E, the Young's Modulus, for all alloy steels at room temperature is about the same and around 30×10^6

$$\epsilon_{spec.} = 73.8 \, \epsilon_{tool} \tag{7.2.9}$$

The results of such calculations are shown in Table 7.2. These expected values can then be compared with the actual strain output as measured with the strain gages and the associated instrumentation.

To measure the actual strain in the specimen groove, 1/64-in. gage length strain gages were mounted in the specimen groove. The method of mounting these gages is shown in Figure 7.8. The results of a limited testing program are shown in Figure 7.9. Specimen 38 in Machine No. 2 gave results within 90 μ in/in of the theoretical, which is quite good. Specimen 79 in Machine No. 1 shows a large discrepancy between observed values and theory. This may be attributed to misalignment of the gages, failure to glue the gages in the exact bottom of the groove, insecure bonding of the gages, or other. Several other specimens were also instrumented and run, but these specimens failed before any useful data could be obtained. These failures are attributed to the fact that the gages, in the specimen groove, are forced to fit a compound curvature. This places severe stresses on the gage-to-carrier bond. Examination of the gages also showed that they were buckling, or "crinkling-up" off of their carriers. For these reasons, this portion of the calibration program was discontinued until satisfactory gages and mounting procedures can be found. Different type gages (1x1M15EC6 foil gage with dynamic leads) have been ordered and will be used to satisfactorily complete this phase of the research.

Calibration of torque bridge.- The torsion strain gage bridge has not as yet been calibrated. Such calibration will be accomplished in the near future.

TABLE 7.2

EXPECTED SPECIMEN STRAIN LEVEL
FOR A GIVEN TOOLHOLDER STRAIN LEVEL

TOOLHOLDER STRAIN (μ in/in)	TOOLHOLDER STRESS (psi)	TOOLHOLDER MOMENT (in-lb)	SPECIMEN STRESS (psi)	SPECIMEN STRAIN (μ in/in)
30	900	575	66,000	2200
35	1050	672	77,200	2570
40	1200	770	88,000	2930
45	1350	864	100,000	3330
50	1500	960	110,000	3670
55	1650	1060	121,000	4030
60	1800	1150	132,000	4400
65	1950	1250	143,500	4780

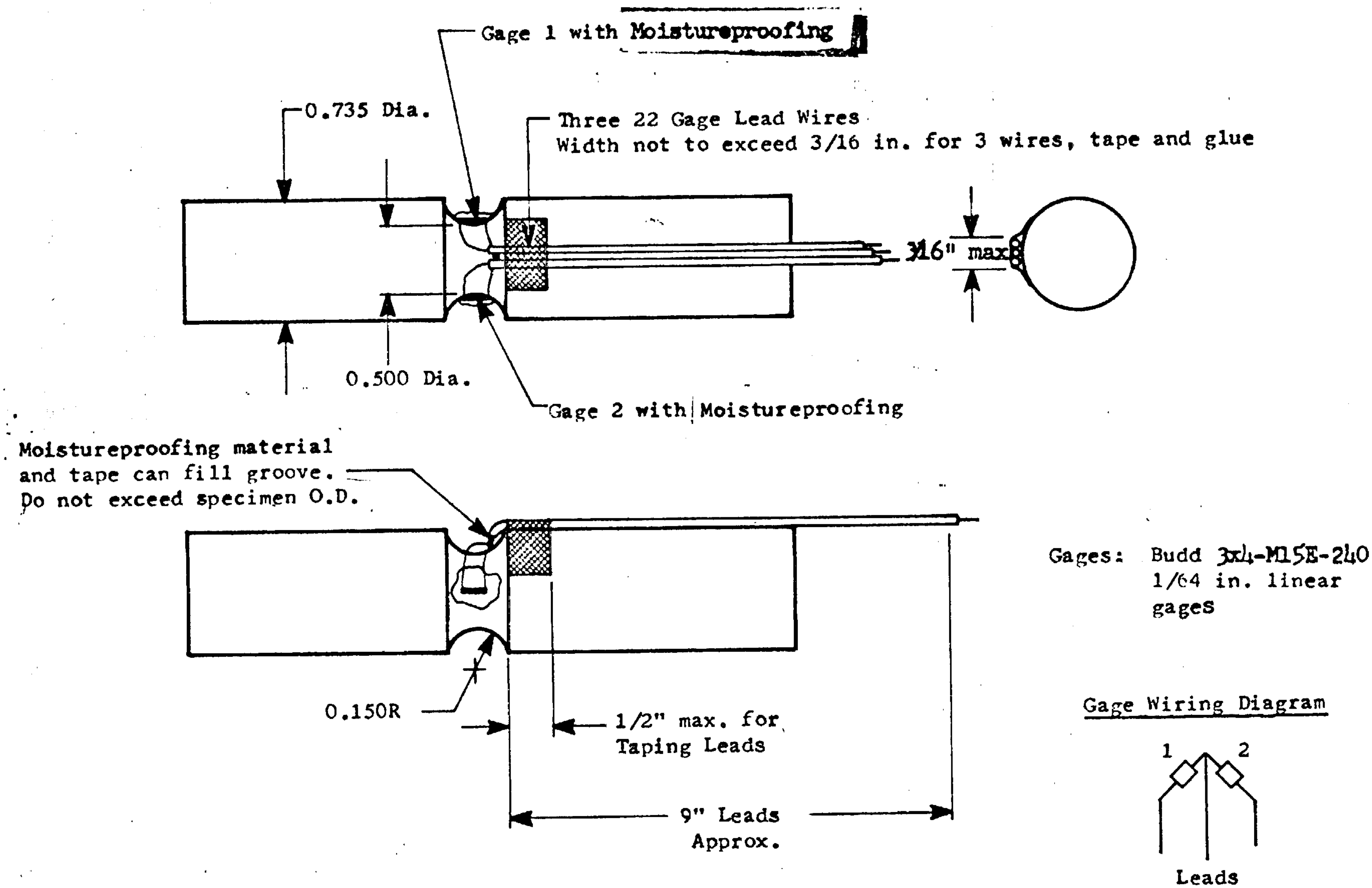


FIGURE 7.8 SKETCH OF SPECIMEN WITH STRAIN GAGES MOUNTED IN THE GROOVE

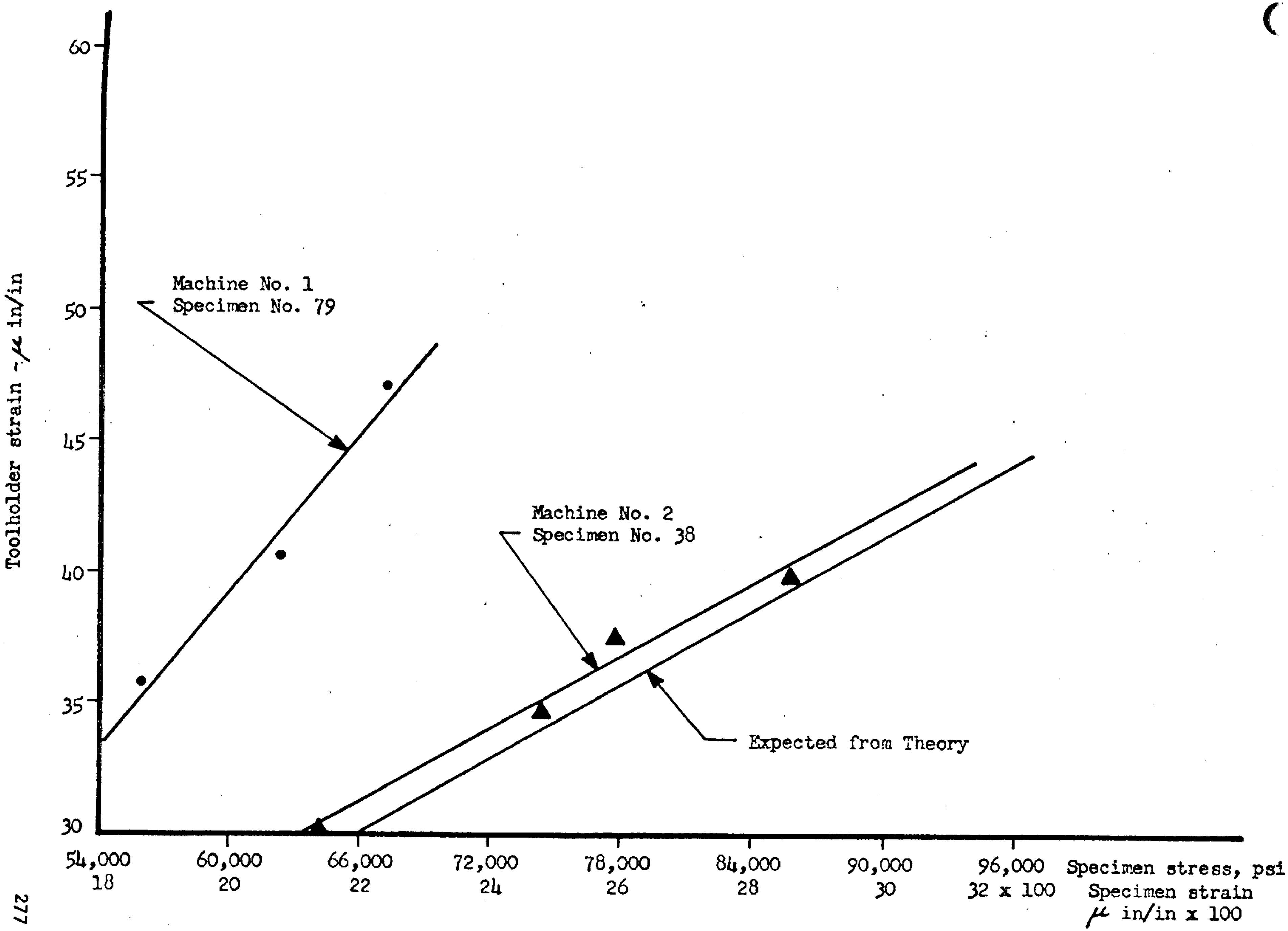


FIGURE 7.9 STRAIN IN SPECIMEN GROOVE VS. STRAIN IN TOOLHOLDER FOR MACHINES NO. 1 AND NO. 2

CHAPTER 7.3

TEST PLAN

Data Requirements

The objective of the testing program is to determine three-dimensional Goodman diagrams such as the one shown in Figure 3.6. To obtain these Goodman diagrams, the testing program described in Chapter 3.3 should be used. Data is needed on life distributions at various stress levels and stress ratios, as previously described.

This testing program is currently in progress at The University of Arizona. A summary of the required data and the degree of completion of the test program indicated is given in Table 7.3. When all of this data is obtained, the three-dimensional Goodman diagrams(at various life cycles) will be known for the test material in its configuration, as described in Chapter 6.4 and Drawing UANASA-6700-B-002. In addition to this specific knowledge, a demonstration of the practicality and applicability of the methodology described in this report will have been accomplished.

The minimum number of specimens required for this program has been estimated in Chapter 3.3 as 648 specimens. This number of specimens will be tested in The University of Arizona's current program. Currently, about 100 specimens have been tested.

Data Sheet

The data sheet used in the testing program is shown in Figure 7.10. The columns in the data sheet are used as follows:

Test Number	The number of the test being conducted is entered serially, beginning with number one and continuing through number 648.
Specimen Serial Number	Each specimen serial number is entered. The order in which the specimens are being tested has been predetermined by a random selection process which will be explained later in this section.

TABLE 7.3

STRESS LEVELS AND STRESS RATIOS*
OF THE UNIVERSITY OF ARIZONA TESTING PROGRAM

Stress Level (psi) \ Stress Ratio	∞	4	2	1	½	¼	0
211,600	P	-	-	-	-	-	Static ultimate tensile strength distribution Minimum of 18 specimens
167,300	C	-	-	-	-	-	
142,700	C	P	-	-	-	-	
118,100	C	P	P	-	-	-	
100,900	P	P	P	P	-	-	
85,000	IP	P	P	P	-	-	
65,000	-	P	P	P	P	-	
50,000	-	P	P	P	P	-	
35,000	-	-	-	P	P	P	
20,000	-	-	-	-	P	P	
10,000	-	-	-	-	P	P	
5,000	-	-	-	-	-	P	

C = Completed IP = In Progress P = Planned

* These stress levels are subject to change based on trends to be indicated by new test results which will be obtained during this research.

DESIGN OF SPECIFIED RELIABILITIES

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FATIGUE DATA

NASA 6700

TEST NUMBER	SPECIMEN SERIAL NO.	STRESS LEVEL		STRESS RATIO	MACHINE NO.	ESTIMATED FAILURE CYCLES	DATE	START TIME	ESTIMATED FAILURE TIME	CYCLES AT FAILURE	VISICORDER RUN	OBSERVER-TEST	REMARKS
		ALT. STRESS (KSI)	MEAN STRESS (KSI)										

ADDITIONAL REMARKS:

FIGURE 7.10 DATA SHEET FOR TESTING PROGRAM

Stress Level	The mean and alternating stresses at which the specimen is being tested are entered.
Stress Ratio	The stress ratio of the alternating stress to the mean stress for the test is entered.
Machine Number	The number of the machine used is entered.
Estimated Failure Cycles	This can be estimated from previous data or theory, and entered.
Date	Date of test is entered.
Start Time	The starting time of the test is entered.
Estimated Failure Time	This can be determined from the estimated cycles to failure and entered.
Cycles at Failure	The actual number of cycles that the specimen ran before failing are recorded here at the end of the test.
Visicorder Run	If a visicorder trace is made for the test, the number of the trace is entered.
Observer Test	The initials of the person conducting the test are entered.
Remarks	Specific observations made by the person conducting the tests are entered, including machine operating condition, excessive specimen vibration, if test acceptable, etc.

This data provides the information required to determine the life distribution by a computer program.

Once a number of these life distributions have been generated, the three-dimensional Goodman diagram can be determined.

Random Sampling

Each test specimen is numbered consecutively when received. The test pieces are then entered in the Specimen Serial No. column in a randomized manner. A book entitled "A Million Random Digits" (3) is used to select the test piece to be tested first, second, etc.. It is advisable to randomize the specimens within one and the same S-N diagram, so that the generated data can be analyzed for that one S-N diagram without having to wait for all S-N diagrams to be developed.

Summary

The test program being conducted at The University of Arizona has been described. The test results obtained to date will be discussed in the next chapter.

CHAPTER 7.4

TEST RESULTS

Digital Computer Programs for Reduction of Data

Programs for life distributions.- Two digital computer programs have been developed for the analysis of cycles-to-failure data at any stress level. These are given in Appendix B. One program computes the mean and standard deviation for the normal distribution, and gives the Chi-square goodness of fit value with its associated degrees of freedom. The other program computes the log mean and log standard deviation for the lognormal distribution fit, and gives the Chi-square goodness of fit value with its associated degrees of freedom. The cycles-to-failure data at each stress level is entered into each program. Based on the results, a decision can be made as to whether the life distributions fit the normal or the lognormal distribution best.

The inputs into each program are the same, and consist of

1. The stress level at which the test was run.
2. The number of specimens tested at this stress level.
3. The number of cycles to failure observed for each specimen, in any order.

The outputs from the computer programs are identical, and consist of

1. Number of specimens.
2. Stress level.
3. Mean (or log mean).
4. Standard deviation (or log standard deviation).
5. Skewness.
6. Kurtosis.
7. Upper and lower limits of the mean (or log mean) and standard deviation (or log standard deviation) at a 95% confidence level.
8. The Chi-square goodness of fit value to the normal (or lognormal) distribution with the associated degrees of freedom.

After the data is processed by these two programs, a decision can be made as to which distribution, normal or lognormal, is most appropriate.

Programs for strength distribution.- Two computer programs also have been developed for the conversion of life distributions to strength distributions for a given stress ratio. This conversion is described in Chapter 3.3 and depicted in Figure 3.5. These programs are given in Appendix C. One program results in a normal strength distribution and the other results in a lognormal strength distribution. The means and standard deviations are given, as well as the Chi-square goodness of fit value with associated degrees of freedom. The strength distributions are computed at various life cycles, the interval used between cycles is at the discretion of the programmer.

The inputs to each program are the same, and consist of

1. The number of life distributions being used.
2. The interpolation step between distributions in psi.
3. The stress levels at which life distributions were taken.
4. The calculated mean (or log mean) and standard deviation (or log standard deviation) for each life distribution.
5. The number of specimens tested at each stress level.
6. The starting point in log cycles for which strength distributions are desired.
7. The increment in log cycles for which strength distributions are required.

The outputs from the programs are identical, and contain

1. The mean (or log mean) and standard deviation (or log standard deviation) of the strength at each life cycle specified by the input.
2. The upper and lower values of the above at the 95% confidence level.
3. The Chi-square goodness of fit value with the associated degrees of freedom for each strength distribution.

Based on the output from these programs, a decision can be made as as to whether normal or lognormal strength distributions best fit the data. Also, three-dimensional Goodman diagrams can now be constructed for specified numbers of life cycles, based on output from the above programs for a number of different stress ratios. Thus the cycle from the procurement of cycles-to-failure data to the generation of three-dimensional Goodman diagrams is completed.

It should be noted in connection with the computer programs for converting life distributions to strength distributions, that the following four combinations are possible:

Life Distribution (input)	Strength Distribution (output)
normal	normal
normal	lognormal
lognormal	normal
lognormal	lognormal

The results of using these programs on the data obtained to date will be discussed next.

Results for Life Distributions

Life distributions have been found for three stress levels for the stress ratio of $s_a/s_m = \infty$, as shown in Table 7.3. The results of the computer runs for these stress levels are shown in Table 7.4 for both the normal and lognormal distribution.

These results can be used to plot an S-N diagram as shown in Figure 7.11. From this figure it can be seen that the slope of the actual S-N diagram is very near to that predicted by the Shigley method as described in Section 6. However, the actual strength levels as determined by test are substantially higher than the theoretical. The following explanations to this apparent difference between theory and experiment were sought:

1. Improper application of the stress concentration factor in the theoretical determination of the endurance strength (Section 6) and in the calibration equation given earlier in this section.

2. Much higher actual strength in a grooved specimen of this material than published data in Reference 60 (Code 1206, p. 1), Section 6, indicates.

The first explanation was explored by reviewing the results in Table 7.2 and Figure 7.9. The specimen stress calculations contain a stress concentration of 1.41 in bending. If this were inapplicable and the calculations were wrong, a significant discrepancy between the theoretical specimen stress and the calibrated stress should have been obtained. In Figure 7.9 the theoretically calculated stress in the specimen groove is larger than that obtained from strain gages, but by only 2,700 psi. This cannot explain the

TABLE 7.4

RESULTS OF COMPUTER RUNS FITTING CYCLES-TO-FAILURE DATA TO NORMAL AND LOGNORMAL DISTRIBUTIONS

Stress Level psi	Normal Distribution						Lognormal Distribution					
	Mean cycles	Standard Deviation cycles	Skewness	Kurtosis	Chi-Square Fit Value	Degrees of Freedom	Log Mean log cycles	Log Standard Deviation log cycles	Skewness	Kurtosis	Chi-Square Fit Value	Degrees of Freedom
167,300	9,029	995	-0.247	1.995	1.382	2	3.952	0.049	-0.407	2.123	2.169	2
142,700	22,171	3,708	-0.042	1.855	0.787	3	4.339	0.074	-0.265	1.945	0.713	2
118,100	77,977	12,195	0.873	3.060	4.637	3	4.887	0.067	0.575	2.696	0.526	3

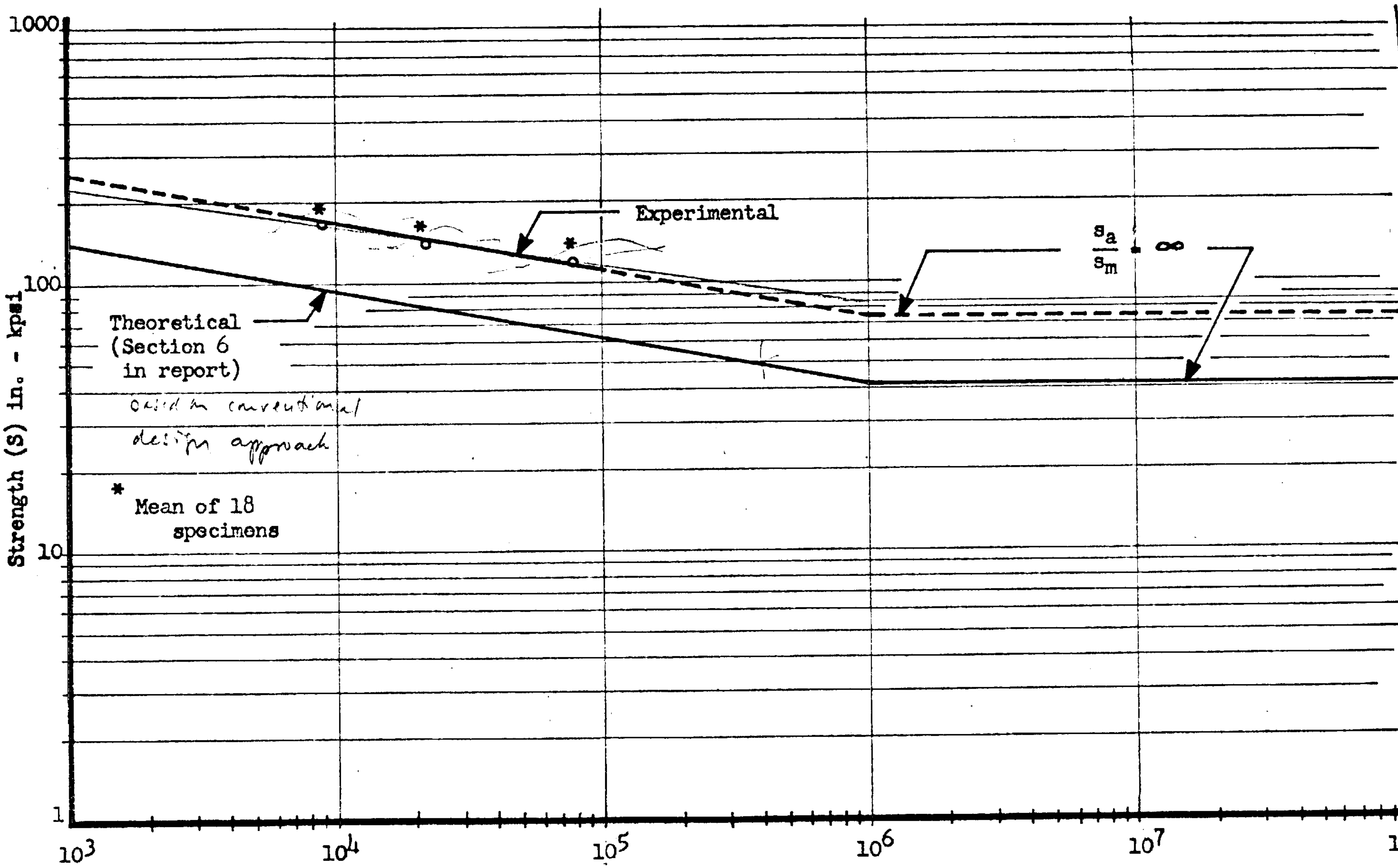

FIGURE 7.11 ESTIMATED THEORETICAL AND EXPERIMENTAL S-N DIAGRAM FOR SAE 4340 STEEL, CONDITION C-4, HEAT TREATED TO R_C 35/40

260,000 - 142,000 = 118,000 psi difference (84% of 142,000), at 10^3 cycles, in the strength curves in Figure 7.9. A more extensive determination of the actual specimen-groove stress by better strain gages will be made to verify the preliminary results of Figure 7.9. If Figure 7.9 is verified, then the validity of the stress concentration factor application at its chosen magnitude will have been ascertained.

The determination of the S-N diagram by the method recommended by Shigley has been accepted by many authorities. There is the possibility, however, that the recommended conversion of the theoretical stress concentration factor, as determined by photoelasticity, to the actual stress concentration factor may not be applicable to this specimen material and geometry under fatigue conditions. The same may apply for the other factors, namely, the size and surface finish factors. This points out the possible inadequacy of the available data on these important design factors for SAE 4340 steel, condition C-4, heat treated to R_C 35/40.

The second explanation may also shed light on the queries raised previously. This aspect was explored by subjecting five specimens to a tensile test. These specimens had a diameter at the base of the groove of 0.499 in. and an area at this section of 0.195 in^2.. The following ultimate loads were obtained: 47,100, 47,900, 48,700, 47,900 and 47,900 lb. with a mean of 47,500 lb. and a range of 1,600 lb.. The ultimate strengths, S_{ult}, should be given by

$$S_{ult} = K_a \frac{P}{A}$$

where

K_a = actual stress concentration factor in direct tension

P = ultimate load, lb.

A = actual sectional area of specimen at ultimate load, in^2.

All five specimens exhibited hardly any necking and a substantially brittle fracture because of the existing groove in the specimens. To get S_{ult}, the actual value of K_a needs to be known. Design books state that for brittle materials subjected to static loading the stress concentration factor can only be determined by experiment, and that theoretical stress concentration factors do not apply. Nevertheless, the actual stress concentration factor should not exceed the theoretical. Consequently, the upper limit of S_{ult} will be that obtained when using the theoretical stress concentration factor, which for the geometry of these test specimens with a $d/D = 0.685$ and $r/D = 0.198$, is 1.60 in direct tension. The lower limit of S_{ult} will be that obtained with a stress concentration factor equal to 1.00. These considerations give the following values for S_{ult}:

$$S_{ult} \text{ maximum} = 1.60 \times \frac{47,500}{0.195}$$

$$= 390,000 \text{ psi}$$

$$S_{ult} \text{ minimum} = 1.00 \times \frac{47,500}{0.195}$$

$$= 243,000 \text{ psi}$$

The S-N diagram's strength value at 10^3 cycles is customarily taken to be 90% of the static ultimate strength. Consequently, the starting strength value of the S-N diagram, S_s, will be between

$$S_s \text{ maximum} = 0.90 \times 390,000 = 350,000 \text{ psi}$$

and

$$S_s \text{ minimum} = 0.90 \times 243,000 = 218,000 \text{ psi}$$

The experimental S-N diagram based on the data obtained so far gives an S_s value of 260,000 psi, which corresponds to an actual stress concentration factor, K_a, of

$$K_a = \frac{S_{ult} \times A}{P} = \frac{S_s \times A}{0.90 \times P}$$

$$= \frac{260,000 \times 0.195}{0.90 \times 47,500}$$

$$= 1.19$$

This value is reasonable and within the range of 1.00 and 1.60 predicted before. The validity of these findings will be ascertained by the second-year effort, during which period more test data will be generated.

Results for Strength Distributions

At the present time, there is not enough test data to enable the conversion of the life distributions to strength distributions. The necessary data is being obtained under the present testing program.

Endurance Limit

The determination of the endurance limit of the specimen requires a slightly different method than that described in Chapter 3.3. Basically, the tests must be concentrated around the endurance limit. The following methods exist for the determination of the endurance limit:

1. Probit method (4, p. 8)
2. Staircase method (4, p. 9)
3. Modified Staircase method (4, p. 10)
4. Step method (4, p. 11)
5. Prot method (4, p. 12)

From these methods the choice was quickly narrowed to the Probit and the Staircase methods (the Modified Staircase method is included as part of the Staircase method) since Method 4 may introduce errors due to the "coaxing" of the material and Method 5 will give more uncertain results than Methods 1, 2 or 3. Table 7.5 shows a comparison of the remaining two methods, with the preferred Staircase method. Actually, since the testing will be split among two machines, the Modified Staircase method will be used.

In the Modified Staircase method a specimen is tested at what is thought to be the endurance limit. If the specimen fails, the stress is lowered one increment and the next specimen is tested. If the specimen does not fail, the stress is increased one increment and the next specimen is tested.

A specimen is a "no-failure" specimen if it runs 10^7 cycles. One increment is 8% of the estimated endurance limit. Analysis of the data is given in reference (4, Section VI).

This method is currently being used at The University of Arizona to obtain the endurance limit for a stress ratio of $s_a/s_m = \infty$. These tests will be conducted for the other stress ratios also.

A discussion of the test results obtained to date is given in the next chapter.

TABLE 7.5

COMPARISON OF TESTING METHODS
FOR ENDURANCE LIMIT

Method	Number of Specimens	Total Time per Endurance Limit	Advantages	Disadvantages	Comments
Probit	50*	52 days (2 mach.)	More accurate	More specimens Longer time	
Staircase	30*	30 days (2 mach.)	Fewer specimens Shorter time Advance knowledge of mean not required		Preferred method

* "A Tentative Guide for Fatigue Testing and the Analysis of Fatigue Data", ASTM, Philadelphia, Pa., 1958, 80 pp.

CHAPTER 7.5

DISCUSSION OF RESULTS

Life Distributions
Normal vs. Lognormal

The results of the Chi-square goodness of fit tests from the testing program to date are summarized in Table 7.6.

These results indicate that the normal distribution is preferred at the 167,300 and 142,700 psi stress levels. The lognormal distribution is preferred at the 118,100 psi stress level. As discussed in Chapter 2.2, it is generally thought that life distributions will be skewed, and thus they should fit the lognormal distribution better than the normal. However, it should also be noted that the higher stress levels approach the area of low-cycle fatigue and static failure. Static strength distributions are usually normal. Therefore, the normal distribution may be the best for tests at higher stress levels. Further testing will provide a better basis for deciding between the normal and lognormal distributions. Also, it would be informative to test more than 18 specimens at each stress level in order to get a more significant Chi-square test.

Strength Distributions
Normal vs. Lognormal

Currently there is no data to support a decision as to whether the strength distributions from this research will be described best by the normal or the lognormal distribution. Generally, they are treated in the literature as normal. In particular, Smith (5) has analyzed large amounts of data on steel wires and has found the normal distribution to give the best fit. Verification of results from this project must await the completion of tests now in progress.

Adequacy of Data and Data Processing Methods

The nature of the data being obtained and the testing program being currently pursued indicate that the desired results outlined in Table 7.3 will be obtained successfully.

The computer programs for data processing have been found to be adequate for constructing the three-dimensional Goodman diagrams which are required for the design by reliability methodology.

The results to date are following expected patterns except for the higher strength exhibited by the specimen material in its present geometry. No major difficulties have been encountered and hopefully none are expected.

TABLE 7.6

SUMMARY OF CHI-SQUARE GOODNESS OF FIT
RESULTS FOR CYCLES-TO-FAILURE DISTRIBUTIONS

Stress Level psi	Distribution	Chi-Square Fit Value	Degrees of Freedom	Level of Significance *
167,300	normal	1.382	2	0.50
167,300	lognormal	2.169	2	0.66
142,700	normal	0.787	3	0.15
142,700	lognormal	0.713	2	0.30
118,100	normal	4.637	3	0.80
118,100	lognormal	0.526	3	0.18

* The level of significance indicates the relative probability of a discrepancy between the test data and the distribution. Therefore, a low level of significance is indicative of a better fit.

CHAPTER 7.6

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The research equipment and instrumentation used in this testing program have been assembled and calibrated and are working satisfactorily. The test plan, described in this section, is now being used to generate the three-dimensional Goodman diagrams which are necessary for the design by reliability methodology.

Although the results generated to date do not justify drawing final conclusions about the data, it can definitely be stated that the program is working and that the capability does exist for obtaining the desired results.

Procedures and programs for reducing data now exist and are quite adequate.

The successful completion of the testing program is merely a matter of time. The necessary data is now being accumulated.

Recommendations

The following recommendations are made regarding the testing program:

1. Complete the current test program as outlined in Table 7.3.
2. Complete the fabrication of test Machine No. 3 in order to obtain the required data faster.
3. Investigate further, by more thorough testing, the nature of the life distributions, and subsequently, the strength distributions.
4. Begin to plan for similar tests to be run on specimens of different materials, or on specimens with different stress concentrations, or on specimens under different environmental conditions, as needed to develop the methodology.

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1. Honeywell Technical Manual: Instructions for Carrier and Linear/Integrate Amplifier Model 119. Denver Division, Denver, Colorado, April 1965.

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3. Staff of Rand Corporation: A Million Random Digits with 100 000 Normal Deviates. The Free Press Publishers, Glencoe, Ill., 1955, 200 pp.

4. ASTM: A Tentative Guide for Fatigue Testing and the Statistical Analysis of Fatigue Data. ASTM Special Technical Publication No. 91-A, 1958, 80 pp.

SECTION 8

OVERALL CONCLUSIONS

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SECTION 8

OVERALL CONCLUSIONS

The design-by-reliability methodology is found to remove the major objection to the conventional non-probabilistic design methodology, whereby not only the geometry and material of a part are established, but also the degree of success of such a design at the design stage, namely its reliability. The design-by-reliability methodology recognizes that each parameter entering the failure-governing stress and the failure-governing strength may be a random variable, and consequently distributed. It follows then that the failure-governing stress and strength are themselves distributed as opposed to conventional design where they are considered as having discrete values. Sufficient tools have been presented in this report to show that the probability of successful function of a designed part can be calculated and thus the true integrity of the designed part determined (Section 1).

In the determination of the failure-governing stress and strength distributions of the various failure-governing strength criteria for alloy steels, the distortion-energy criterion is found to apply best in case of fatigue with the maximum shear stress criterion a secondary alternative. Nevertheless, to determine the quantitative reliabilities precisely enough substantial refinements are necessary in the determination of the true failure-governing strength distributions, be they based on the distortion energy or the maximum shear stress theories of failure. One refinement is the requirement that failure-governing strength surfaces be determined rather than simplifications such as in the form of modified Goodman diagrams. Such simplifications distort the failure-governing strength picture and, unfortunately, drive the design further away from the optimum usually, though not necessarily always, towards the over-designed direction (Section 1).

A great scarcity of data on the distributions of stress and strength parameters is found to exist today. Major efforts need to be expended for their acquisition (Section 1).

Another major need in the determination of the failure-governing stress and strength distributions is the synthesis of the distributions of the parameters involved into these distributions, once the functional relationships between the failure-governing stress and strength and their parameters are established. It is found that there are seven promising analytical methods for accomplishing such syntheses, namely:

1. The Algebra of Normal Functions

2. Change of Variable

Preceding page blank

3. Moment Generating Function

4. Fourier Transform, Convolution, and Inversion

5. Mellin Transform, Convolution, and Inversion

6. Characteristic Function

7. Cumulative Distribution Function

The conditions of applicability of these methods indicate that the majority of the required syntheses can be achieved; however, much additional analytical research is required to establish the true distribution of the dependent variable when the independent variables are non-normal. Even in the instance of normal variables, exact analytical methods for arriving at the distribution of the dependent variable are not available when the independent variables are in a functional form other than in terms of sums or differences. This is also true when the dependent variables are not totally independent of each other (Section 2).

The work which has been done for the synthesis of specific distributions in specific functional relationships has been summarized in Tables 2.1 and 2.2 (Section 2).

Also, numerical methods exist for the synthesis of distributions. The Fourier and Mellin transform methods, for example, lend themselves to numerical solution; whereas, the Monte Carlo method is a direct numerical approach to all problems of synthesis of distributions, thus making it a very powerful method (Section 2).

The Algebra of Normal Functions method appears to provide the most expeditious method for determining the failure-governing stress and strength distributions from normally distributed random variables. However, exact solutions are obtained only for sums and differences of random variables, and the remaining cases are only approximations. Only in instances where the variances of the distributions are a small percentage of their means and the distributions are substantially removed from the origin does The Algebra of Normal Functions method provide results of sufficient accuracy. This aspect requires further investigation (Section 2).

Only meager published statistical information exists on the distributions of parameters entering the failure-governing stress function. This dearth of information includes such important design parameters as loads, stress concentration, notch sensitivity, and such physical properties as Young's Modulus, Poisson's Ratio and others (Section 3).

The same picture is found for the parameters entering the failure-governing strength function. Among these parameters are the following: size, surface finish, manufacturing processes, surface treatment, corrosion, cavitation, wear, and temperature. Hardly any distributional data on these factors has been found to exist. This again points out to the fact that much research needs to be conducted for the generation of such much-needed design data so that the design-by-reliability methodology can be implemented with its full potential (Section 3).

In the instance of fatigue strength distribution determination, computer methods were developed for the determination of life cycles-to-failure distributions at specified stress levels and for their conversion to strength distributions at specific life cycles. These methods enable the design of parts subjected to fatigue loads on the basis of their reliability (Sections 3 and 4).

Methods have been presented for the analytical, semi-analytical, and computer determination of the reliability of a mechanical component given its failure-governing stress and strength distributions, regardless of the nature of these distributions (Section 5).

A major effort for the first year's segment of this research was expended in the design, construction, and development of a combined bending-torsion fatigue reliability research machine as no such commercially available machines could be found. These machines utilize the four-square principle, enabling the locking-in of the torsional stresses which are superimposed onto the reversed bending stresses in a rotating specimen simulating a shaft. These machines are unique, enabling the application of 3,540 in.-lb. of bending moment and 5,400 in.-lb. of torque. A specimen can be subjected to either torsional or bending stresses, or both, over a wide range of stress ratios of bending stress to torsional stress (Section 6).

Two of these machines are presently running satisfactorily in a research program for the determination of the failure-governing strength surface for approximately 750 specimens made of SAE 4340 cold-drawn steel, condition C-4, heat treated to hardness R_C 35/40. These specimens have a 3/4 in. major diameter, and are grooved to a 1/2 in. minor diameter, with a groove radius of 0.145 in.. This results in a theoretical stress concentration factor of 1.41. The test plan involves six stress levels and six stress ratios to enable the generation of experimental fatigue strength surfaces which will enable the designer to couple with his fatigue failure-governing stress distribution surface and calculate the component reliability, when the component is subjected to such stresses (Section 6).

The fatigue research program being pursued with these machines is 11% complete and the major effort is presently being expended in the generation of more fatigue data towards the completion of the planned experimental research program. The results obtained to date are given in Table 7.3. A preliminary reduction of the test data obtained to date has been accomplished and the result summarized in Table 7.6. This reduction involves the fitting of the normal and lognormal distributions to the life cycles-to-failure data. Conclusions as to whether the normal or lognormal distribution best fits this data has to await the completion of the research program and the determination of the strength distributions involved (Section 7).

It should be noted that relatively higher strengths have been exhibited by the specimen material in its present geometry than found in the published literature and handbooks, indicating the necessity for research into this aspect for an adequate explanation of this phenomenon.

The successful completion of the experimental research program, of the determination of the time-dependent strength distributions, and of the theoretical and computer determination of the resulting reliabilities for the material being researched upon is merely a matter of time.

SECTION 9

OVERALL RECOMMENDATIONS

SECTION 9

OVERALL RECOMMENDATIONS

The design-by-reliability methodology should be used to the fullest extent possible by the modern design engineer. However, many aspects remain to be explored further as the field is still in its early stages of development. Much more research needs to be conducted in this field, including the following:

1. The true failure-governing strength criteria and their statistical applicability to specific materials and load combinations should be determined, including those of the distortion energy and maximum shear stress.

2. The applicable failure-governing strength surfaces should be experimentally determined, and the theories developed in Item 1 above should be verified statistically.

3. The distributions of strength design factors, including those of size, surface finish, manufacturing processes, surface treatment, corrosion, cavitation, wear, and temperature, should be determined.

4. The failure-governing stress factor distributions, including those of loads, stress concentration, notch sensitivity, and of such physical properties as Young's Modulus, Poisson's Ratio, and others, should be determined.

5. The Fourier Transform and Mellin Transform methods should be developed more fully to extend their usefulness for the synthesis of functions of random variables for mechanical reliability applications.

6. Analytical and computer methods for the synthesis of the following distributions should be developed:

   1. Weibull
   2. Gamma
   3. Beta

   These distributions are of potential importance in the area of mechanical reliability.

7. Other functions of random variables such as $Y = \ln X$, $Y = e^X$, $Y = \sin X$, etc., should be investigated. These will eventually be necessary for mechanical reliability work.

8. Other distributions should be investigated further, including the following:

1. Series Representation
2. Extremal
3. Double Exponential
4. Lattice

The applicability of these distributions to mechanical reliability should be studied.

9. Further efforts should be expended to determine the applicability of the various distributions on a physical and phenomenological basis rather than on the empirical basis of best-fit-to-data.

10. The question of what is an independent and what is a dependent random variable in mechanical reliability theory should be examined closely and care should be taken to apply the results of functions of random variables correctly, based on whether the variables are truely independent or dependent.

11. The functions of random variables for mixed (not identically distributed) distributions, such as the product of lognormal and normal distributions, should be studied. The results of such studies, as found in the literature, do not appear, as yet, to be too applicable to mechanical reliability.

12. The accuracy of the approximations used in The Algebra of Normal Functions should be evaluated.

13. The Monte Carlo method should be developed more extensively for solving problems involving functions of random variables.

14. Analytical and computer methods should be developed for the determination of the reliability of a complex part when the failure-governing stress and strength distributions are surfaces.

15. The strength surfaces for specimens of different geometry and stress concentrations than those being presently tested should be determined.

16. The strength surfaces for specimens of materials different than those being presently tested should be determined.

17. The strength surfaces for specimens subjected to a variety of other actual environmental conditions should be determined.

18. The phenomenon of the relatively higher strength exhibited by the grooved SAE 4340 specimens used in this research and the indication that notched specimens exhibit higher apparent strengths than unnotched specimens should be thoroughly investigated.

APPENDIX A

TYPICAL MONTE CARLO PROGRAM LISTING

```
      JINX = 16
      DIMENSION XMEAN(5), XSIG(5), XPT(5), XFREQ(5), CHI1(5),
     1      YPT(2001), PROP(2001), Y(35), OFRQ1(5,10)
     2  ,OFREQ(34), YMID(34), RAN(2002,2), YPROB(34)
      EQUIVALENCE (RAN,YPT) , (RAN(2002),PROP)
    5 READ 10
   10 FORMAT (80H                    NONE
     1                                     )
      READ 20,N,NRAND
   20 FORMAT(2I5)
      READ 30, (XMEAN(I),XSIG(I),I=1,N)
   30 FORMAT ((2F20.0))
      PRINT 50 ,(XMEAN(I),XSIG(I),I=1,N)
   50 FORMAT (1H1,27H INPUT VARIABLE PARAMETERS.,23X,
     1    9H      MEAN,8X,12HSTANDARD DEV/(45X,2F20.6))
      DO 40  I=1,JINX
      RAN(I) = XRANF(2)
   40 CONTINUE
      RANGE = 8.0
      RAND = NRAND
      Z = RANGE/100.0
      NDFG = 9
      NRAND = NRAND+1
   60 CONTINUE
      DO 70  J=1,N
      CHI1(J) = 0.0
      DO 70 K=1,10
      OFRQ1(J,K) = 0.0
   70 CONTINUE
      EFREQ = RAND/10.0
      DO 71  I=1,NRAND
      DO 71  J=1,N
      RAN(I,J) = XRANF(2)
      RAN(I,J) = RAN(I,J)+1.0
   71 CONTINUE
C     CONFIDENCE CHECK OF RANDOM VARIABLES
      NM1 = NRAND-1
      DO 75  J=1,N
      L = 0
      XJ = 0.
      DO 75 K=1,10
      L = L+1
      XJ = XJ +10.
      DO 75  I=1,NM1
      IF(RAN(I,J)-XJ) 73,73,75
   73 IF(RAN(I,J)-(XJ-10.)) 75,74,74
   74 OFRQ1(J,L) = OFRQ1(J,L)+1.0
   75 CONTINUE
      DO 76  J=1,N
      DO 76 K=1,10
      CHI1(J) = CHI1(J)+(OFRQ1(J,K)-EFREQ)*(OFRQ1(J,K)-
     1    EFREQ)/EFREQ
```

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```
   76 CONTINUE
      PRINT 150
  150 FORMAT (1H0,32HCONFIDENCE LEVEL CHECK OF RANDOM,
     1     10H VARIABLES)
      DO 190 J=1,N
      PRINT 170 ,(CFRG1(J,K),K=1,10)
      PRINT 180 ,NDEG,CHI1(J)
  170 FORMAT(14H RANDOM NUMBER, F9.1,9F10.1)
  180 FORMAT(22H DEGREES OF FREEDOM = ,I2,12H  CHI SQUARE,
     1     9H VALUE = ,F10.4, 18H FOR RANDOM NUMBER)
  190 CONTINUE
      DO 210 L=1,N
      IF(CHI1(L)-6.40)  210,210,60
  210 CONTINUE
C     RANDOM GENERATION OF INPUT DISTRIBUTION VALUES
      DO130 I=1,NRAND
      DO 80 J=1,N
      ZPT = Z*RAN(I,J)-4.0
      EXP = (ZPT*ZPT*0.5)
      XFREQ(J) = 0.39894228*EXPEF(-EXP)*RANGE/100.
      XPT(J) = XMEAN(J)+ZPT*XSIG(J)
   80 CONTINUE
C     RESULTANT DISTRIBUTION POINTS AND PROB GENERATION
      YPTI = (44.8*XPT(1))/((3.14159)*(XPT(2)**3.))
      YPT(I) = YPTI
      PROB(I) = 1.0
      DO 90 J=1,N
      PROB(I) = PROB(I)*XFREQ(J)
   90 CONTINUE
  130 CONTINUE
      NRAND = NRAND-1
C     RESULTANT DIST HISTOGRAM, MEAN, STD DEV AND OTHER
C        IMPORTANT PARAMETERS
      MCL = 2.0+2.3*LOGF(RAND)
      YMAX = YPT(1)
      YMIN = YPT(1)
      DO 260 I=2,NRAND
      IF (YPT(I)-YMIN)220,230,230
  220 YMIN = YPT(I)
      GO TO 250
  230 IF (YPT(I)-YMAX)250,250,240
  240 YMAX = YPT(I)
  250 CONTINUE
  260 CONTINUE
      RANGE = YMAX-YMIN
      YCL = MCL
      WIDTH = RANGE/YCL
      NCL = MCL+1
      Y(1) = YMIN
      DO 270 I=2,NCL
      IM1 = I-1
      Y(I) = Y(IM1)+WIDTH
```

```
      YMID(IM1) = Y(IM1)+WIDTH/2.0
  270 CONTINUE
      DO 310 I=2,NCL
      IM1 = I-1
      CFREQ(IM1) = 0.0
      YCUM = 0.0
      YPROB(IM1) = 0.0
      DO 310 K=1,NRAND
      IF (YPT(K)-Y(IM1)) 300,280,280
  280 IF(YPT(K)-Y(I)) 290,300,300
  290 CFREQ(IM1) = CFREQ(IM1)+1.0
      YPROB(IM1) = YPROB(IM1)+PROB(K)
  300 CONTINUE
  310 CONTINUE
      FI = 0.0
      FIUI = 0.0
      DO 320 K=1,NRAND
      FI = FI+PROB(K)
      FIUI = FIUI+PROB(K)*YPT(K)
  320 CONTINUE
      YM1 = FIUI/FI
      Y2ND1 = 0.0
      Y3RD1 = 0.0
      Y4TH1 = 0.0
      DO 330 K=1,NRAND
      SQ = (YPT(K)-YM1)*(YPT(K)-YM1)
      Y2ND1 = Y2ND1+SQ*PROB(K)
      Y3RD1 = Y3RD1+(YPT(K)-YM1)*PROB(K)*SQ
      Y4TH1 = Y4TH1+SQ*SQ*PROB(K)
  330 CONTINUE
      Y2ND1 = Y2ND1/FI
      Y3RD1 = Y3RD1/FI
      Y4TH1 = Y4TH1/FI
      IF(Y2ND1) 332,334,334
  332 PRINT 333,Y2ND1
  333 FORMAT (E20.8)
      GO TO 5
  334 CONTINUE
      YSIG1 = SQRTF(Y2ND1)
      YK3 = Y3RD1/(YSIG1*Y2ND1)
      YK4    = Y4TH1/(Y2ND1*Y2ND1)
      FI = 0.0
      FIUI = 0.0
      DO 340 K=1,NCL
      FI = FI+YPROB(K)
      FIUI = FIUI+YPROB(K)*YMID(K)
  340 CONTINUE
      YM2 = FIUI/FI
      PRINT 350,YM1
  350 FORMAT(1H0,36HDIST MEAN BY SUMMATION OF INDIVIDUAL,
     1   14H PROBABILITIES,F15.4)
      PRINT 360,YM2
```

```
360 FORMAT (1X,31HDIST MEAN BY SUMMATION OF CLASS,
   1    19H MIDPOINT FREQUENCY,F15.4)
    PRINT 10
    PRINT 370
370 FORMAT (1X,12HNO OF VALUES)
    PRINT 380
380 FORMAT (1X,15H GENERATED,10X,13H DIST MEAN, 7X,
   1  13HSTD DEVIATION,12X,8HSKEWNESS,12X,8HKURTOSIS)
    PRINT390,NRAND,YM1,YSIG1,YK3,YK4
390 FORMAT (1X,I10,2F20.4,2F20.8)
    PRINT 400
400 FORMAT (1H0,31HHISTOGRAM CLASS INTERVAL VALUES)
    PRINT 410
410 FORMAT (1X,14HCLASS INTERVAL, 8X,12HOBSERVATIONS, 9X,
   1    11HPROBABILITY,16X, 9HMID POINT, 13X,
   2    12HBEGINNING PT)
    PRINT420,(I, OFREQ(I),YPROB(I),YMID(I),Y(I),I=1,MCL)
420 FORMAT (7X,I3, 5X, F20.1,  F20.8,2F25.8)
    SPROB = 0.0
    DO 430 I=1,MCL
    SPROB = SPROB+YPROB(I)
430 CONTINUE
    DO 440 I=1,MCL
    YPROB(I) = YPROB(I)/SPROB
440 CONTINUE
    PRINT 450
450 FORMAT (36H NORMALIZED HISTOGRAM CLASS INTERVAL,
   1    7H VALUES)
    PRINT 460
460 FORMAT (1X,14HCLASS INTERVAL, 9X,11HPROBABILITY, 7X,
   1    13HAVERAGE POINT)
    PRINT 470, (I,YPROB(I),YMID(I), I=1,MCL)
470 FORMAT ((7X,I3,5X,2F20.8))
    GO TO 5
    END
```

APPENDIX B

TYPICAL PROGRAM LISTING FOR LIFE DISTRIBUTIONS

```
      SUBROUTINE NIFIT(N, XN, XMEAN, XSIG, CHISQ,MCL)
C     CHI SQUARED FIT OF DATA POINTS TO A NORMAL DIST
   46 FORMAT(1X,11HEXPONENT = ,F5.0,19H CLASS MIDPOINT = ,
     1    F5.3)
   47 FORMAT (19H DATA POINT VALUES/(1X,F11.3,9F12.3))
   48 FORMAT (11X,10H EXPECTED ,10X,10H OBSERVED ,10X,
     1    7H  CLASS/14X,10HCLASS FREQ,10X,10HCLASS FREQ,
     2    8X,12HBEGINNING PT/(1X,F20.0,F20.1,F20.8))
      DIMENSION XN(120), X(15), XMID(14), OFREQ(15),
     1    EFREQ(15)
      AN = N
      CLASS = 2.0+3.3*LOGF(AN)
      MCL = CLASS
      XMAX = XN(1)
      XMIN = XN(1)
      DO 60 I=2,N
      IF(XN(I)-XMIN) 52,54,54
   52 XMIN = XN(I)
      GO TO 58
   54 IF(XN(I)-XMAX) 58,58,56
   56 XMAX = XN(I)
   58 CONTINUE
   60 CONTINUE
      RANGE = XMAX-XMIN
      XCL = MCL
      WIDTH = RANGE/XCL
      NCL = CLASS+1.0
      X(1) = XMIN
      DO 70 I=2,NCL
      IM1 = I-1
      X(I) = X(IM1)+WIDTH
      XMID(IM1) = X(IM1)+WIDTH/2.0
   70 CONTINUE
      DO 80 I=2,NCL
      IM1 = I-1
      OFREQ(IM1) = 0.0
      DO 76 J=1,N
      IF(XN(J)-X(IM1)) 76,72,72
   72 IF(XN(J)-X(I)) 74,76,76
   74 OFREQ(IM1) = OFREQ(IM1)+1.0
   76 CONTINUE
   80 CONTINUE
      OFREQ(MCL) = OFREQ(MCL)+1.0
      DO 83 I=1,MCL
      EXP = ((XMID(I)-XMEAN)*(XMID(I)-XMEAN))/(2.0*XSIG*XSIG)
      IF (EXP-100.) 82,82,81
   81 EFREQ(I) = 0.0
      PRINT 46, EXP,XMID(I)
      GO TO 83
   82 CONTINUE
      EFREQ(I) = AN*0.3989423/XSIG*EXPFF(-EXP)*WIDTH
   83 CONTINUE
```


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```
      PRINT 47,(XM(I),I=1,M)
      PRINT 48, (EFREQ(I),OFREQ(I),X(I),I=1,MCL)
      M = MCL/2
      J = 1
      DO 84 I=1,M
      IF (EFREQ(I)-4.0) 77,84,84
   77 IF (I-M) 79,84,84
   79 J=J+1
      IP1 = I+1
      EFREQ(IP1) = EFREQ(I)+EFREQ(IP1)
      OFREQ(IP1) = OFREQ(I)+OFREQ(IP1)
      EFREQ(I)=0.0
      OFREQ(I)=0.0
   84 CONTINUE
      K = MCL
      MCTR = MCL-M+1
      DO 87 I=1,M
      MMI = MCL-I+1
      IF (EFREQ(MMI)-4.0) 85,87,87
   85 IF (MMI-MCTR) 87,87,86
   86 K = MCL-I
      MMI = MMI-1
      EFREQ(MMI) = EFREQ(MMI)+EFREQ(MMI+1)
      OFREQ(MMI) = OFREQ(MMI)+OFREQ(MMI+1)
      EFREQ(MMI+1)=0.0
      OFREQ(MMI+1)=0.0
   87 CONTINUE
      PRINT 48, (EFREQ(I),OFREQ(I),X(I),I=J,K)
      MCL = K-J
      CHISQ = 0.0
      DO 88 I=J,K
      IF(EFREQ(I))100,88,100
  100 CONTINUE
      CHISQ = CHISQ+((OFREQ(I)-EFREQ(I))*(OFREQ(I)-EFREQ(I)
     1   ))/EFREQ(I)
   88 CONTINUE
      RETURN
      END

      PROGRAM FOR FINDING THE MEAN, STANDARD DEVIATION
        AND CONFIDENCE LIMITS OF RANDOM VARIABLES FROM
        ASSUMED NORMAL DISTRIBUTIONS
      CONF = 90.
    2 FORMAT (I5)
    5 FORMAT (F10.0)
    6 FORMAT ((8F10.0))
    1 FORMAT (F10.0,I5/(8F10.0))
   12 FORMAT (1H0// 10H SPECIMENS, 7X,8HSTRENGTH, 12X,
     1   8HLOG MEAN,19X,11HLOG STD DEV,7X,8HSKEWNESS,7X,
     2   8HKURTOSIS/1X,I10,F15.2,F20.8,10H  WITH M =,
     3   F20.8,2F15.3)
   13 FORMAT (1X,  20HCONFIDENCE LIMITS OF,I3,
```

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```
    1    17H PERCENT ARE USED)
 14 FORMAT (1X, 17HCONFIDENCE LIMITS,10HMEAN LOWER,5X,
    1    10HMEAN UPPER,5X,13HSTD DEV LOWER,5X,
    2    13HSTD DEV UPPER/3X,F20.8,F15.4,F18.8,F18.8)
 15 FORMAT (41X, 10HWITH N-1 =,F20.8)
 16 FORMAT (1X, 32H CHI SQUARE FIT TO NORMAL DIST =, F14.8,
    1    5X,20HDEGREE OF FREEDOM = ,I3)
    DIMENSION XN(120),XNLOG(120),CHIL(100),CHIU(100),T(100)
    READ 3,CONF
    READ 6,(T(I),I=1,30)
    READ 5,  AREA
    READ 6, (CHIL(I),I=1,100)
    READ 6, (CHIU(I),I=1,100)
 13 READ 10, S,N,(XN(I),I=1,N)
    DO 20 I=1,N
    XNLOG(I) = LOGF(XN(I))
 20 CONTINUE
    XNSUM = 0.0
    DO 25 I=1,N
    XNSUM = XNSUM + XNLOG(I)
 25 CONTINUE
    TN = N
    AN = N
    XMLOG = XNSUM/AN
    XM2 = 0.0
    XM3 = 0.0
    XM4 = 0.0
    DO 30 I=1,N
    XSQ = (XNLOG(I)-XMLOG)*(XNLOG(I)-XMLOG)
    XM2 = XM2+XSQ
    XM3 =XM3 + XSQ*(XNLOG(I)-XMLOG)
    XM4 = XM4+XSQ*XSQ
 30 CONTINUE
    XSIG = SQRTF(XM2/(AN-1.0))
    XM2 = XM2/AN
    XM3 = XM3/AN
    XM4 = XM4/AN
    XSIGL = SQRTF(XM2)
    XK3 = XM3/(XSIGL*XM2)
    XK4 = XM4/(XM2*XM2)
    A = SQRTF(AN)
    IF (N-30) 34,34,36
 34 XML = XMLOG - T(N-1)*XSIGL/A
    XMU = XMLOG+T(N-1)*XSIGL/A
    GO TO 38
 36 XML = XMLOG-AREA*XSIGL/A
    XMU = XMLOG+AREA*XSIGL/A
    IF (N-100) 37,37,35
 35 N = 100
    AN = N
 37 CONTINUE
 38 XSTDL = XSIGL*SQRTF((AN-1.0)/CHIL(N-1))
```

```
      XSTDU = XSIGL*SQRTF((AN-1.0)/CHI(N-1))
      N = TN
      PRINT 12, N,S,XMLOG,XSIGL, XK3,XK4
      PRINT 15, XSIG
      PRINT 13, CONF
      PRINT 14, XML,XMU,XSTDL,XSTDU
      IF (N-10) 40,40,39
   39 CONTINUE
      CALL KIFIT(N,XMLOG,XMLOG,XSIGL,CHISQ,MCL)
      PRINT 16,CHISQ,MCL
   40 CONTINUE
      GO TO 18
      END
```

```
    95
6.314       2.920       2.353       2.132       2.015       1.943       1.895
1.833       1.812       1.796       1.782       1.771       1.761       1.753
1.740       1.734       1.729       1.725       1.721       1.717       1.714
1.708       1.706       1.703       1.701       1.699       1.697
1.645
3.84146     5.99147     7.81473     9.48773     11.0705     12.5916     14.0671
16.9190     18.3070     19.6751     21.0261     22.3621     23.6848     24.9958
27.5817     28.8693     30.1435     31.4104     32.6705     33.9244     35.1725
37.6525     38.8852     40.1133     41.3372     42.5569     43.7729     45.0
47.4        48.6        49.8        51.0        52.2        53.4        54.6
56.9        58.1        59.3        60.5        61.7        62.8        64.0
66.3        67.5        68.7        69.9        71.0        72.2        73.3
75.6        76.8        77.9        79.1        80.2        81.4        82.5
84.8        86.0        87.1        88.3        89.4        90.5        91.7
93.9        95.1        96.2        97.4        98.5        99.6        100.7
103.0       104.1       105.3       106.4       107.5       108.6       109.8
112.0       113.1       114.3       115.4       116.5       117.6       118.8
121.0       122.1       123.2       124.3
.00393214   .102587     .351846     .710721     1.145476    1.63539     2.16735
3.32511     3.94030     4.57481     5.22603     5.89186     6.57063     7.26094
8.67176     9.39046     10.1170     10.8508     11.5913     12.3380     13.0905
14.6114     15.3791     16.1513     16.9279     17.7083     18.4926     19.3
20.9        21.7        22.5        23.3        24.1        24.9        25.7
27.3        28.1        29.0        29.8        30.6        31.4        32.3
33.9        34.8        35.6        36.4        37.3        38.1        39.0
40.6        41.5        42.3        43.2        44.0        44.9        45.7
47.4        48.3        49.2        50.0        50.9        51.7        52.6
54.3        55.2        56.1        56.9        57.8        58.7        59.5
61.3        62.1        63.0        63.9        64.7        65.6        66.5
68.2        69.1        70.0        70.9        71.8        72.6        73.5
75.2        76.2        77.0        77.9
```

APPENDIX C

TYPICAL PROGRAM LISTING FOR STRENGTH DISTRIBUTIONS

```
      SUBROUTINE PROB(Z,PROB)
C     PROB = THE AREA UNDER NORMAL DISTRIBUTION BETWEEN
C        PLUS AND MINUS Z STANDARD DEVIATIONS
      IF(Z-1.2) 1000,1000,1010
 1000 ZSQ = Z*Z
      PROB = 0.79788455*Z*(0.99999774-ZSQ*(0.16659433
     1  -ZSQ*(0.02468331 - ZSQ*0.0023974867)))
      GO TO 1070
 1010 IF(Z-2.9) 1020,1060,1060
 1020 ZSQ = Z*Z
      PROB = 1.0
      PTERM = 1.0
      FACT = 1.0
      ODDIN = 3.0
 1030 PTERM = -PTERM*ZSQ/(2.0*FACT)
      TERM = PTERM/ODDIN
      PROB = PROB + TERM
      IF (ABSF(TERM) - 0.00007) 1050,1040,1040
 1040 FACT = FACT + 1.0
      ODDIN = ODDIN + 2.0
      GO TO 1030
 1050 PROB = 0.79788453*Z*PROB
      GO TO 1070
 1060 REC = 1./(Z*Z)
      PROB = 1.-0.79788453*EXPEF(-Z*Z/2.0)/Z*
     1   (1. - REC*(1. - REC*(3. - REC*(15.-REC*105.))))
 1070 CONTINUE
      RETURN
      END

      N = 23
      INT = 10
      DIMENSION ASTR(10), AMLOG(10), ASIGL(10), SLAM(9),
     1   XMLOG(251), XSIGL(251), AREA(251,23), FREQ(251,23),
     2   CYC(23), SMLOG(23), SSIGL(23), SK3(23), SK4(23),
     3 NUM(10),INDEX(10),OFREQ(10,23),GFREQ(10,23),
     4   CHISQ(10,23),ST(10),SLSIG(9), EFREQ(10,23)
      EQUIVALENCE (AREA,FREQ),
     1   (XSIGL,SSIGL), (XMLOG,OFREQ), (AREA(500),EFREQ),
     2   (AREA,CHISQ), (AREA(250),GFREQ)
    5 READ10,XJMIN,XJINT
   10 FORMAT(2F10.0)
C     FAILURE DISTRIBUTION VALUES ARE READ IN FROM
C        LOWEST TO HIGHEST STRESS AND STRESSES ARE
C        INTEGER VALUES OF SINT
      READ 20
   20 FORMAT  (50H              BLANK
      READ 30, N,SINT,(ASTR(J),AMLOG(J),ASIGL(J),J=1,N)
   30 FORMAT (I5,F5. /(3F10.0))
      READ32,(NUM(J),J=1,N)
   32 FORMAT ((8I10))
      NM1 = N-1
```


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```
      L = 1
      INDEX(1) = 1
      DO 85 J=1,NM1
      DSTR = ASTR(J+1)-ASTR(J)
      DMEAN = AMLOG(J+1)-AMLOG(J)
      DSIG = ASIGL(J+1)-ASIGL(J)
      SLAM(J) = DSTR/DMEAN
      IF(DSIG) 40,50,40
   4  SLSIG(J) = DSTR/DSIG
   5  CONTINUE
      JP1 = J+1
      MX = DSTR/SINT
      INDEX(JP1) = L+MX
      SMIN = ASTR(1)
      XMLOG(1) = AMLOG(1)
      XSIGL(1) = ASIGL(1)
      DO 80 K=1,MX
      LX = L + K
      XMLOG(LX) = XMLOG(LX-1)+SINT/SLAM(J)
      IF(DSIG) 70,60,70
   6  XSIGL(LX) = XSIGL(LX-1)
      GO TO 80
   7  XSIGL(LX) = XSIGL(LX-1)+SINT/SLSIG(J)
      CONTINUE
      L = LX
   85 CONTINUE
      PRINT 90
      FORMAT (1H1, 8X,12HSTRENGTH-PSI,11X,8HLOG MEAN, 9X,
     1   11HLOG STD DEV,9X,11HINTEGER (I))
      XINT = INT
      STR = SMIN
      DO110 I=1,LX,INT
      PRINT100,STR,XMLOG(I),XSIGL(I),I
      FORMAT ((3E20.8,I20))
      STR = STR+SINT*XINT
  11  CONTINUE
      NOLD = N
      N = LX
C     CONVERTING LOGNORMAL FAILURE DIST. PARAMETERS AT N
C        STRESS LEVELS TO CUMMULATIVE LOGNORMAL FAILURE
C        DISTRIBUTION
      DO 150 I=1,N
      XVAL = XJMIN-XJINT
      DO 1  J=1,M
      XVAL = XVAL+XJINT
      Z = (XVAL-XMLOG(I))/XSIGL(I)
      IF(Z) 120,140,140
      IF(Z+2.5)125,130,130
      Z = -Z
      CALL PROB(Z,PROB)
      F (I,J) = (1.0-PROB)/2.0
      GO TO 1
```

```
140 IF(Z-3.5) 150,150,170
150 CALL PROB(Z,PROB)
    AREA(I,J) = PROB/2.0+0.5
    GO TO 180
160 AREA(I,J) = 0.0
    GO TO 180
170 AREA(I,J) = 1.0
180 CONTINUE
    INT2 = INT*4
    STR = SMIN
    DO 220 I=1,N,INT2
    PRINT 190,STR,XMLOG(I),XSIGL(I)
190 FORMAT(1H ,5X,11HSTRLNGTH = ,F13.6,5X, 9HLOG MEAN ,
   1   9HCYCLES = ,F8.6,5X,20HLOG STD DEVIATION = ,F8.6)
    PRINT 200
200  FORMAT (1X, 33HDATA BELOW IS J, AND CUMMULATIVE ,
   1   13HDIST UP TO J.)
    PRINT 210, (J, AREA(I,J),J=1,M)
210 FORMAT    ((1X,I3,F9.6,I3,F9.6,I3,F9.6,I3,F9.6,
   1  I3,F9.6,I3,F9.6,I3,F9.6,I3,F9.6,I3,F9.6))
    STR = STR+SINT*XINT*4.
220 CONTINUE
    DO 222 J=1,NOLD
    I = INDEX(J)
    XNUM = NUM(J)
    DO 224 K=1,M
    CFREQ(J,K) = XNUM*AREA(I,K)
224 CONTINUE
222 CONTINUE
C     FREQUENCY DIST AND NORMAL DISTRIBUTION PARAMETERS
    XJMIN = XJMIN-XJINT
    DO 260 J=1,M
    XJ = J
    CYC(J) = XJ*XJINT+XJMIN
    PRINT 240, J, CYC(J)
240 FORMAT (1H0,4HJ = ,I3 ,5X,13HLOG CYCLES = ,F6.3)
    PRINT 250, (I,AREA(I,J),I=1,N,INT)
250 FORMAT (2X,2H(I),1X,9HFREQUENCY/(1X,I4,F10.6,I5,F10.6
   1  ,I5,F10.6,I5,F10.6,I5,F10.6,I5,F10.6,I5,F10.6,I5,F10.6))
260 CONTINUE
    NM1 = N-1
    DO 270 J=1,M
    DO 270 I=1,NM1
    IP1 = I+1
    FREQ(I,J) = AREA(IP1,J) -AREA(I,J)
270 CONTINUE
    N = NM1
C     MEAN AND NORMAL DISTRIBUTION PARAMETERS OF HISTOGRAM
    DO 300 J=1,M
    FI = 0.
    FIUI = 0.
    UI = SMIN-SINT*0.5
```

```
    DO 280 I=1,N
    UI = UI+SINT
    FI = FI+FREQ(I,J)
    FIUI = FIUI+FREQ(I,J)*UI
280 CONTINUE
    SMLOG(J) = FIUI/FI
    SM2 = 0.
    SM3 = 0.
    SM4 = 0.0
    UI = SMIN-SINT*0.5
    DO 290 I=1,N
    UI = UI+SINT
    SQ = (UI-SMLOG(J))*(UI-SMLOG(J))
    SM2 = SM2 + SQ*FREQ(I,J)
    SM3 = SM3 + FREQ(I,J)*(UI-SMLOG(J))*SQ
    SM4 = SM4+SQ*SQ*FREQ(I,J)
290 CONTINUE
    SM2 = SM2/FI
    SM3 = SM3/FI
    SM4 = SM4/FI
    SSIGL(J) = SQRTF(SM2)
    SK3(J) = SM3/(SSIGL(J)*SM2)
    SK4(J) = SM4/(S 12*SM2)
300 CONTINUE
    PRINT 20
    PRINT 310
310 FORMAT (10H  NUMBER   ,10HLOG CYCLES,5X,
   1    15H  MEAN STRENGTH,5X,15H STD. DEVIATION,10X,
   2    10H  SKEWNESS,10X,10H  KURTOSIS)
    PRINT 32-, (J, CYC(J), SMLOG(J), SSIGL(J), SK3(J),
   1    SK4(J), J=1,N)
    PRINT 320, (J, CYC(J), SMLOG(J), SSIGL(J), SK3(J), SK4(J), J=1,N
320 FORMAT ((I8,F9.2,2F24.4,2F20.8))
    PRINT 325,(NUM(J),INDEX(J),J=1,NOLD)
325 FORMAT ( 15X, 15HOBSERVED NUMBER, 9X, 11HINTEGER (I)/
   1    (10X,2I20))
    DO 500 K=1,N
    DO 480 J=1,NOLD
    X = INDEX(J)
    ST(J)  = SMIN+SINT*X -SINT
    Z = (ST(J)-SMLOG(K))/SSIGL(K)
    IF(Z) 330,350,350
330 Z = -Z
    IF (Z-3.5) 340,360,360
340 CALL PROB(Z,PROB)
    PROBA = 0.5-PROB*0.5
    GO TO 370
350 IF(Z-3.5) 355,355,365
355 CALL PROB(Z,PROB)
    PROBA = PROB*0.5+0.5
    GO TO 370
360 PROBA = 0.0
```

```
      GO TO 270
  265 PROBA = 1.0
  270 GFREQ(J,K) = PROBA
      XNUM = NUM(J)
      EFREQ(J,K) = GFREQ(J,K)*XNUM
      CHISQ(J,K) = (OFREQ(J,K)-EFREQ(J,K))*(OFREQ(J,K)
     1    -EFREQ(J,K))/EFREQ(J,K)
  480 CONTINUE
      PRINT 490,K
  490 FORMAT (1H0,3H K=,I3)
      PRINT 495,(ST(J),GFREQ(J,K),OFREQ(J,K),EFREQ(J,K),
     1    CHISQ(J,K),J=1,NOLD)
  495 FORMAT(16X,14HSTRENGTH LEVEL, 9X,11HNORMAL FREQ, 7X,
     1    13HOBSERVED FREQ, 7X,13HEXPECTED FREQ, 4X,
     2    16HCHI SQUARE VALUE/ (10X,5F20.6))
  500 CONTINUE
      GO TO 5
      END
```

APPENDIX D

LIST OF DRAWINGS
FOR
TEST MACHINE

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Drawing No.	Title
UANASA-6700-B-001	TEST SPECIMEN
UANASA-6700-B-002	TEST SPECIMEN
UANASA-6700-D-003	TEST MACHINE
UANASA-6700-D-003	TEST MACHINE
UANASA-6700-D-004	LOADING FRAME
UANASA-6700-A-005	LINK PIN
UANASA-6700-C-006	BEARING HOUSING
UANASA-6700-C-007	INDEXER FLANGE
UANASA-6700-B-008	BACK SHAFT
UANASA-6700-D-009	BASE PLATE
UANASA-6700-D-010	MOUNTS
UANASA-6700-B-011	LOADING LINK
UANASA-6700-B-012	FULCRUM UNIT
UANASA-6700-B-013	HORIZONTAL LINK
UANASA-6700-B-014	VERTICAL LINK
UANASA-6700-A-015	RETAINER PLATE
UANASA-6700-B-016	LOADING LINK
UANASA-6700-A-017	FULCRUM BLOCK
UANASA-6700-A-018	RETAINER PLATE
UANASA-6700-A-019	SLIP RING BUSHING
UANASA-6700-B-020	LOADING BAR ASSEMBLY
UANASA-6700-A-021	LINK PIN
UANASA-6700-A-022	MODIFICATION 30XL037 PULLEY

Drawing No.	Title
UANASA-6700-B-023	MECHANICAL COUNTER MOUNT
UANASA-6700-B-024	SLIP RING BRUSH MOUNT
UANASA-6700-A-025	COUNTER BUSHING
UANASA-6700-B-026	BRIDGE
UANASA-6700-B-027	BEARING SHIELD CLAMP
UANASA-6700-B-028	BRIDGE POST
UANASA-6700-C-029	INSTRUMENTATION POST ARM
UANASA-6700-B-030	INSTRUMENTATION POST
UANASA-6700-B-031	BEARING SHIELDS
UANASA-6700-A-032	SPARE BELT BRACKET

ABSTRACT

A PROBABILISTIC METHOD OF DESIGNING SPECIFIED RELIABILITIES INTO MECHANICAL COMPONENTS WITH TIME DEPENDENT STRESS AND STRENGTH DISTRIBUTIONS

By Dr. Dimitri Kececioglu, Joe W. McKinley and Maurice J. Saroni
The University of Arizona
January, 1967, 330 pp.

A basic methodology for design-by-reliability in combined-stress fatigue, with time dependent strength distributions, is developed and discussed. Numerous examples are given, utilizing both the Von Mises and maximum shear stress theories of failure. Mathematical methods used in treating problems in functions of random variables are thoroughly reviewed, and results which apply to functions of random variables are given. Methods discussed include: Algebra of Normal Functions, Change of Variable, Moment Generating Function, Fourier Transform, Mellin Transform, Characteristic Function, Cumulative Distribution, and Monte Carlo. Methods for determining and handling failure-governing stress and strength distributions are given. Methods for determining reliability once the failure-governing stress and strength are known are given.

The design, fabrication, and operation of a combined-stress fatigue testing machine for reliability research are discussed. A testing program for verification of the design-by-reliability methodology is described.